Department of Economics

Princeton University

ECO S500 (First Half). Fall 2005.

Suggested Answers to Homework 3

1. (a) If X has full rank (i.e, rank $= k$), then $X'X$ also has rank k and therefore is invertible. In this case, we know that the orthogonal projection of any vector $y \in \mathbb{R}^n$ on to col (X) is given by:

$$P_{\text{col}(X)}(y) = X(X'X)^{-1}X'y$$

Therefore $P_{\text{col}(X)}(y)$ is a linear transformation. It will be diagonalizable if and only if the projection matrix $X(X'X)^{-1}X'$ has k linearly independent eigenvectors.

(b) In this case, we have

$$X'X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

which has full rank equal to 2. The projection matrix onto col (X) is:

$$X(X'X)^{-1}X' = \begin{bmatrix} \frac{1}{1.2} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{1.2} \end{bmatrix}$$

The projections for y_1 and y_2 are given by:

$$P_{\text{col}(X)}(y_1) = X(X'X)^{-1}X'y_1 = \begin{bmatrix} \frac{1}{1.2} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{1.2} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0 \\ 0.5 \end{bmatrix}$$

$$P_{\text{col}(X)}(y_2) = X(X'X)^{-1}X'y_2 = \begin{bmatrix} \frac{1}{1.2} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{1.2} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So the projection of y_2 on to col (X) is simply y_2. This shows that y_2 is in fact an element of col (X). This is not true for y_1. We can express the projections in terms of the vectors $\widehat{\beta}_1$, $\widehat{\beta}_2$ where $P_{\text{col}(X)}(y_1) = X\widehat{\beta}_1$; $P_{\text{col}(X)}(y_2) = X\widehat{\beta}_2$. These vectors of coefficients are

$$\widehat{\beta}_1 = (X'X)^{-1}X'y_1 = \begin{bmatrix} -1 \\ 0.5 \end{bmatrix} \quad ; \quad \widehat{\beta}_2 = (X'X)^{-1}X'y_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Notice that

$$y_2 = -2 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \; ; \quad \text{and therefore } y_2 \in \text{col } (X)$$

2. We have

$$T_1[e_1] = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \; ; \quad T_1[e_2] = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \; ; \quad T_1[e_3] = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \; \Rightarrow S_{T_1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Diagonalizability is a property of symmetric matrices. Therefore it does <u>not apply to S_{T_1}</u>. For T_2 we have:

$$T_2[e_1] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \; ; \quad T_2[e_2] = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \; ; \quad T_2[e_3] = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \; \Rightarrow S_{T_2} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Since S_{T_2} is an upper triangular matrix, the characteristic equation is simply:

$$\left| S_{T_2} - \lambda I_2 \right| = 0 \;\Leftrightarrow\; (1-\lambda)(2-\lambda)(1-\lambda) = 0$$

So this matrix has only two eigenvalues, $\lambda_1 = 1$ and $\lambda_2 = 2$. The eigenvectors that correspond to λ_1 must satisfy

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

This is satisfied only if $p_2 = p_3 = 0$. Therefore, all eigenvectors for λ_1 must be of the form

$$\begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}$$

for some $\alpha \in \mathbb{R}$. We're in trouble: There is no way of choosing to such vectors that are linearly independent. The eigenvectors for λ_2 must satisfy:

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 2p_1 \\ 2p_2 \\ 2p_3 \end{bmatrix}$$

This is satisfied only if $p_3 = 0$ and $p_1 = -p_2$. Therefore, all eigenvectors of λ_2 have the form

$$\begin{bmatrix} \gamma \\ -\gamma \\ 0 \end{bmatrix}$$

for some $\gamma \in \mathbb{R}$. It is impossible to find three linearly independent vectors from any collection of vectors that look like

$$\begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} \quad \text{and/or} \quad \begin{bmatrix} \gamma \\ -\gamma \\ 0 \end{bmatrix}$$

S_{T_2} is not diagonalizable by any matrix, complex or real.

3. The proposed transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ is not a linear transformation: The last element of the transformation is always one. Therefore, the matrix representation does not apply to it. The kernel of T is the empty set $\emptyset$, since no value of x, y, z will yield a vector of zeros. We will always have 1 at the last entry. Therefore, the nullity of T, which is the dimension of its kernel is equal to zero. This is the answer to the problem. However, in order to put in practice all we know about linear transformations, consider the following modification of T:

$$T[(x, y, z)] = (x - y + z, 3x - 5y + 2z, x - 3y, 0)$$

(change the one for a zero). Now we have something susceptible to the properties we know about linear transformations. In particular, there exists a 4×3 matrix S_T that represents T:

$$T[e_1] = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}; \quad T[e_2] = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 0 \end{bmatrix}; \quad T[e_3] = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow S_T = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -5 & 2 \\ 1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The reduced-row echelon form of this matrix is:

$$\text{r.r.e.f } (S_T) = \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & 0.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \text{ rank } (S_T) = 2$$

since $\text{rank } (S_T) = \dim \big(\text{col } (S_T)\big)$, the Dimension Theorem yields $3 = \text{rank } (S_T) + \dim \big(\text{null } (S_T)\big) \Rightarrow \dim \big(\text{null } (S_T)\big) = 3 - \text{rank } (S_T) = 3 - 2 = 1$. Since $\dim \big(\text{null } (S_T)\big) =$

dim $\big(\ker\,(T)\big)$ = nullity of T, we have that the nullity of T (the dimension of its kernel space) is one.

4. The proposed transformation has the form

$$T\big(a + bx + cx^2\big) = ax + bx^2 + cx^3.$$

We use the basis $B_{P_2} = \{1, x, x^2\}$ and $B_{P_3} = \{1, x, x^2, x^3\}$. We find the matrix S_T in the following way:

$$\begin{array}{c} \\ 1 \rightarrow \\ x \rightarrow \\ x^2 \rightarrow \\ x^3 \rightarrow \end{array} \quad \begin{array}{c} T(1) \\ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \end{array} \quad \begin{array}{c} T(x) \\ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \end{array} \quad \begin{array}{c} T(x^2) \\ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \end{array} \quad \Rightarrow S_T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that for any $p(x) = ax + bx^2 + cx^3$, we have

$$\begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} 0 \\ a \\ b \\ c \end{bmatrix} = ax + bx^2 + cx^3 \checkmark$$

5. First note that this system has the property that $x_1 = Ax_0$, $x_2 = A \cdot x_1 = A \cdot Ax_0$, $x_3 = A \cdot x_2 = A \cdot A \cdot Ax_0$, …, therefore $x_n = A^n x_0$ for each n. If the matrix A is diagonalizable, then $A^2 = P^{-1}DP \cdot P^{-1}DP = P^{-1}D^2P$, $A^3 = P^{-1}DP \cdot P^{-1}D^2P = P^{-1}D^3P$ and in general, $A^n = P^{-1}D^nP$. We can see that the vector x_n will converge to a finite vector only if D^n converges to a finite matrix. But the matrix D is the diagonal matrix that includes the eigenvalues of A in its main diagonal. D^n has zeros outside its diagonal, which consists of the terms λ_1^n, $\lambda_2^n, \ldots, \lambda_k^n$. A well-defined norm of D^n is given by $\|D^n\|_\infty = \max\big\{|\lambda_1^n|, |\lambda_2^n|, \ldots, |\lambda_k^n|\big\}$. This norm will have a finite limit only if $\max\big\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_k|\big\} \leq 1$, where $|\lambda|$ denotes the modulus of λ.

6. Denote $u \equiv x^2 + x^3$. First, we identify a basis for P_1. As usual, we choose the simplest one: $B_{P_1} = \{1, x\}$. Next, we identify how the projection looks like: Since we are projecting on to P_1, we must have $P(u) = \widehat{\beta}_0 + \widehat{\beta}_1 x$. The problem consists of finding $\widehat{\beta}_0$, $\widehat{\beta}_1$. Next, we express the orthogonality conditions that define the projection: The "residual" $u - \widehat{\beta}_0 - \widehat{\beta}_1 x$ must be

orthogonal (using the inner product defined above) to each one of the elements on the basis B_{P_1}. Therefore, $\widehat{\beta}_0$ and $\widehat{\beta}_1$ must satisfy:

$$\left\langle x^2 + x^3 - \widehat{\beta}_0 - \widehat{\beta}_1 x, 1 \right\rangle = 0; \quad \left\langle x^2 + x^3 - \widehat{\beta}_0 - \widehat{\beta}_1 x, x \right\rangle = 0$$

$$\Rightarrow \quad \int_a^b \left(x^2 + x^3 - \widehat{\beta}_0 - \widehat{\beta}_1 x\right) dx = 0; \quad \int_a^b \left(x^2 + x^3 - \widehat{\beta}_0 - \widehat{\beta}_1 x\right) \cdot x dx = 0$$

So, we have a system of two equations and two unknowns ($\widehat{\beta}_0$ and $\widehat{\beta}_1$). Solving this system yields:

$$\widehat{\beta}_0 = \frac{1}{30}\left[-6a^3 - 4ab(5+6b) - b^2(5+6b) - a^2(5+24b)\right]$$
$$\widehat{\beta}_1 = a + \frac{9a^2}{10} + b + \frac{6ab}{5} + \frac{9b^2}{10}.$$

We can also find the orthogonal projection as the solution to a problem of "distance minimization". We know that whenever a vector space has an inner product, we can construct a well-defined norm as $\| \cdot \| = \sqrt{< \cdot, \cdot >}$. Therefore, we can represent this projection as the solution to:

$$\min_{\widehat{\beta}_0, \widehat{\beta}_1} \left\| x^2 + x^3 - \widehat{\beta}_0 - \widehat{\beta}_1 x \right\| = \min_{\widehat{\beta}_0, \widehat{\beta}_1} \sqrt{\int_a^b \left(x^2 + x^3 - \widehat{\beta}_0 - \widehat{\beta}_1 x\right)^2 dx} = \min_{\widehat{\beta}_0, \widehat{\beta}_1} \int_a^b \left(x^2 + x^3 - \widehat{\beta}_0 - \widehat{\beta}_1 x\right)^2 dx$$

The first order conditions for this problem are exactly the system of orthogonality conditions we used above. Therefore, the solution is the same.

7. Once again, we start by identifying a basis of the subspace onto which we are projecting. Choose the basis $B_{P_1} = \{1, x\}$. The projection has to be a function of the form $\widehat{\beta}_0 + \widehat{\beta}_1 x$, and the following orthogonality conditions must hold:

$$\left\langle 1 + x^2 - \widehat{\beta}_0 - \widehat{\beta}_1 x, 1 \right\rangle = 0; \quad \left\langle 1 + x^2 - \widehat{\beta}_0 - \widehat{\beta}_1 x, x \right\rangle = 0$$

Given the inner product we are using, these conditions become:

$$\left(1 + 2^2 - \widehat{\beta}_0 - \widehat{\beta}_1 \cdot 2\right) \cdot 1 + \left(1 + 3^2 - \widehat{\beta}_0 - \widehat{\beta}_1 \cdot 3\right) \cdot 1 + \left(1 + 5^2 - \widehat{\beta}_0 - \widehat{\beta}_1 \cdot 5\right) \cdot 1 = 0$$
$$\left(1 + 2^2 - \widehat{\beta}_0 - \widehat{\beta}_1 \cdot 2\right) \cdot 2 + \left(1 + 3^2 - \widehat{\beta}_0 - \widehat{\beta}_1 \cdot 3\right) \cdot 3 + \left(1 + 5^2 - \widehat{\beta}_0 - \widehat{\beta}_1 \cdot 5\right) \cdot 5 = 0$$

This system simplifies to:

$$41 - 3\widehat{\beta}_0 - 10\widehat{\beta}_1 = 0$$

$$170 - 10\widehat{\beta}_0 - 38\widehat{\beta}_1 = 0$$

The solution becomes $\widehat{\beta}_0 = -\frac{71}{7}$, $\widehat{\beta}_1 = \frac{50}{7}$.

8. Consider first the partial derivatives:

$$\lim_{\alpha\to 0}\frac{f(\alpha,0)-f(0,0)}{\alpha}=\lim_{\alpha\to 0}\frac{\sqrt{|\alpha\cdot 0|}-0}{\alpha}=\lim_{\alpha\to 0}0=0$$
$$\lim_{\alpha\to 0}\frac{f(0,\alpha)-f(0,0)}{\alpha}=\lim_{\alpha\to 0}\frac{\sqrt{|0\cdot\alpha|}-0}{\alpha}=\lim_{\alpha\to 0}0=0$$

Therefore, both partial derivatives exist at 0, and they are equal to zero. Now suppose we take any other directional derivative. These would satisfy $u_1\neq 0$ and $u_2\neq 0$. In this case we have:

$$\lim_{\alpha\to 0}\frac{f(\alpha u_1,\alpha u_2)-f(0,0)}{\alpha}=\sqrt{\left|\alpha^2u_1u_2\right|}\alpha=\frac{\left|\alpha\right|}{\alpha}\cdot\sqrt{|u_1u_2|}$$

We have

$$\lim_{\alpha\to 0^+}\frac{\left|\alpha\right|}{\alpha}\cdot\sqrt{|u_1u_2|}=\sqrt{|u_1u_2|};\quad \lim_{\alpha\to 0^-}\frac{\left|\alpha\right|}{\alpha}\cdot\sqrt{|u_1u_2|}=-\sqrt{|u_1u_2|}$$

since both limits differ, then

$$\lim_{\alpha\to 0}\frac{\left|\alpha\right|}{\alpha}\cdot\sqrt{|u_1u_2|}$$

does not exist. Consequently, no directional derivative other than the partial derivatives exist at $(0,0)$.

HW 3 Solutions to Questions 5 & 6

Note Title 9/13/2005

(5) Taylor Series

(a) $f_1(x,y) = x^{\frac{1}{4}} y^{3/4}$

Denote (x,y) the 1×2 matrix or row vector

$$\begin{pmatrix} \frac{\partial f_1}{\partial x} \\ \frac{\partial f_1}{\partial y} \end{pmatrix} = \nabla f_1(x,y) = \begin{pmatrix} y^{\frac{3}{4}} \cdot \frac{1}{4} x^{-3/4} \\ x^{\frac{1}{4}} \cdot \frac{3}{4} y^{-1/4} \end{pmatrix}$$

$$H_{f_1}(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y} \end{pmatrix} = \begin{pmatrix} y^{3/4}\left(\frac{-3}{16}\right) x^{-7/4} & x^{-3/4} \frac{3}{16} y^{-1/4} \\ x^{-3/4} \frac{3}{16} y^{-1/4} & x^{\frac{1}{4}}\left(\frac{-3}{16}\right) y^{-5/4} \end{pmatrix}$$

Taylor Series approximation of order 2 around (x_0, y_0) is

$$f(x,y) = f(x_0,y_0) + (x-x_0, y-y_0) \cdot \nabla f_1(x_0,y_0) + (x-x_0, y-y_0) H_{f_1}(x_0,y_0)(x-x_0, y-y_0)$$

<Let $(x_0,y_0) = (1,1)$>

$$f(x,y) = 1 + (x-1, y-1)\begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix} + (x-1, y-1)\begin{pmatrix} -3/16 & 3/16 \\ 3/16 & -3/16 \end{pmatrix}\begin{pmatrix} x-1 \\ y-1 \end{pmatrix}$$

The Hessian is symmetric by Young's Theorem, & any symmetric matrix has real eigenvalues. Since the Hessian is also real, & the system defining eigenvectors from eigenvalues is linear, we can find real eigenvectors. Hence it is diagonalizable by a real matrix, since a symmetric matrix can always be diagonalized.

(b) $f_2(x,y) = x^2 + y^2 + e^{xy} + \sin^2 x$

$$\nabla f_2(x,y) = \begin{pmatrix} 2x + ye^{xy} + 2\sin x \cos x \\ 2y + xe^{xy} \end{pmatrix}$$

$$H_{f_2}(x,y) = \begin{pmatrix} 2 + y^2 e^{xy} + 2\cos^2 x - 2\sin^2 x & yxe^{xy} + e^{xy} \\ xye^{xy} + e^{xy} & 2 + x^2 e^{xy} \end{pmatrix}$$

$$\nabla f_2(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad H_{f_2}(0,0) = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$$

$f(0,0) = 1$

so 2^{nd} order Taylor approximation is (around $(0,0)$):

$$f_2(x,y) = 1 + (x,y)\begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

And, $H_{f_2}(0,0)$ is diagonalizable by a real matrix for the same reason as in part (a).

(c) $f_3(x,y,z) = x\sin z - z\sin y$

$$\nabla f_3(x,y,z) = \begin{pmatrix} \sin z \\ -z\cos y \\ x\cos z - \sin y \end{pmatrix}$$

$$H_{f_3}(x,y,z) = \begin{pmatrix} 0 & 0 & \cos z \\ 0 & z\sin y & -\cos y \\ \cos z & -\cos y & -x\sin z \end{pmatrix}$$

$f_3(-1, \frac{\pi}{2}, 0) = -\sin(0) - 0 = 0$

$$\nabla f_3(-1, \tfrac{\pi}{2}, 0) = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$$

$$H_{f_3}(-1, \tfrac{\pi}{2}, 0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

So the 2^{nd} order Taylor approx. of f_3 around $(-1, \pi/2, 0)$ is

$$f(x,y,z) = -2z + (x+1, y-\pi/2, z)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x+1 \\ y-\pi/2 \\ z \end{pmatrix}$$

& $H_{f_3}(-1, \frac{\pi}{2}, 0)$ is diagonalizable by a real matrix for the same reason as before.

(6) Implicit Function

(a) $x^2 - xy^3 + y^5 - 17 = 0$

Let $F(x,y) = x^2 - xy^3 + y^5 - 17$. Note $F(5,2) = 0$.

$$\nabla F(x,y) = \begin{pmatrix} 2x - y^3 \\ -3xy^2 + 5y^4 \end{pmatrix}$$

At $(5,2)$, $\nabla F(5,2) = \begin{pmatrix} 2 \\ 20 \end{pmatrix}$

By implicit fn. theorem, since $\frac{\partial F}{\partial y} \neq 0$

$$y'(x)\Big|_{(5,2)} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}\Bigg|_{(5,2)} = \frac{-2x + y^3}{-3xy^2 + 5y^4}\Bigg|_{(5,2)} = \frac{-2}{20} = \frac{-1}{10}$$

(b) $F(x,y) = x^2 - 3xy + y^3 - 7 = 0$. Note $F(4,3) = 0$.

$$\nabla F(x,y)\Big|_{(4,3)} = \begin{pmatrix} 2x - 3y \\ -3x + 3y^2 \end{pmatrix}\Bigg|_{(4,3)} = \begin{pmatrix} -1 \\ 15 \end{pmatrix}.$$

By implicit fn. thm, since $\frac{\partial F}{\partial y} \neq 0$

$$y'(x)\Big|_{(4,3)} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}\Bigg|_{(4,3)} = \frac{-2x + 3y}{-3x + 3y^2}\Bigg|_{(4,3)} = \frac{1}{15}$$

(c) $F(x,y,z) = x^3 + 3y^2 + 4xz^2 - 3z^2y - 1 = 0$

(i) $(x,y) = 1,1$

Then $F(1,1,z) = \cancel{1} + 3 + 4z^2 - 3z^2 - \cancel{1} = 0$

or $z^2 = -3$

Then need $z = \pm\sqrt{3}\, i$

$$\nabla F(x,y,z)\Big|_{(1,1,z)} = \begin{pmatrix} 3x^2 + 4z^2 \\ 6y - 3z^2 \\ 8xz - 6zy \end{pmatrix}\Bigg|_{(1,1,z)} = \begin{pmatrix} 3 + 4z^2 \\ 6 - 3z^2 \\ 2z \end{pmatrix}$$

Note that at $z = \pm\sqrt{3}\, i$, $\frac{\partial F}{\partial z}(1,1,\pm\sqrt{3}\, i) \neq 0$

we don't allow $F: \mathbb{C} \to \mathbb{C}$, done.

(ii) If $(x,y) = (1,0)$ then $F(1,0,z) = 1 + 4z - 1 = 0$
or need $z = 0$

$$= \begin{pmatrix} 3x^2 + 4z^2 \\ 6y - 3z^2 \\ 8xz - 6zy \end{pmatrix}\Bigg|_{(1,0,0)} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

cannot use imp. fn thm. to find z as fn. of x, y since $\frac{\partial F}{\partial z} = 0$

(iii) $(x,y) = (\frac{1}{2}, 0)$

$F(\frac{1}{2}, 0, z) = \frac{1}{8} + 2z^2 - 1 = 0$
or $2z^2 = \frac{7}{8}$ or $z = \pm \frac{\sqrt{7}}{4}$

For $\frac{\sqrt{7}}{4}$:

$$\begin{pmatrix} 3x^2 + 4z^2 \\ 6y - 3z^2 \\ 8xz - 6zy \end{pmatrix}\Bigg|_{(\frac{1}{2}, 0, \frac{\sqrt{7}}{4})} = \begin{pmatrix} 5/2 \\ -\frac{21}{16} \\ \sqrt{7} \end{pmatrix}$$

$\frac{\partial F}{\partial z} \neq 0$,

so $\frac{\partial z(x,y)}{\partial y} = \frac{\frac{21}{16}}{\sqrt{7}}$, $\frac{\partial z(x,y)}{\partial x} = \frac{-5/2}{\sqrt{7}}$

similarly for $\frac{-\sqrt{7}}{4}$

(d) $F(x,y,z) = x^2 - y^2 + z^2 = 0$

(i) at $(x,y) = (1,1)$, $F(1,1,z) = z^2 = 0 \Rightarrow z = 0$.

$\nabla F = \begin{pmatrix} 2x \\ -2y \\ 2z \end{pmatrix}$ so at $z = 0$, $\frac{\partial F}{\partial z} = 0$, can't use thm.

(ii) at $(x,y) = (6,3)$, $F(6,3,z) = 36 - 9 + z^2 = 0 \Rightarrow 27 = -z^2$ or ~~$z = \pm 3i\sqrt{3}$~~
forget complex z

7. (a) Take an orthonormal basis $\epsilon_1, \epsilon_2, ...\epsilon_n$ of V of which $\epsilon_1, \epsilon_2, ..., \epsilon_m$ is an orthonormal basis of W. Let $(x_{1i}, x_{2i}, ..., x_{ni})$ be the coordinates of e_i with respect to the above basis and $y_{ki} = \frac{x_{ki}}{di}$. Then

$$\begin{vmatrix} y_{11} & y_{21} & \cdots & y_{n,1} \\ y_{12} & y_{22} & \cdots & y_{n,2} \\ \cdots & \cdots & \cdots & \cdots \\ y_{1,n} & y_{2,n} & \cdots & y_{n,n} \end{vmatrix}$$

is an orthogonal matrix (remember that $e_1, e_2, ..., e_n$ form an orthogonal basis of V)

If the length of the projection of e_i is equal to d for any i:

$$y_{1i}^2 + y_{2i}^2 + ...y_{mi}^2 = (x_{1i}^2 + x_{2i}^2 + ... + x_{mi}^2)d_i^{-2} = d^2 d_i^{-2} \tag{1}$$

Since $d \leq d_i$:

$$m = \sum_{k=1}^{m}\sum_{i=1}^{n} y_{ki}^2 \ (\{\tfrac{e_i}{d_i}\} \text{ is an orthonormal basis})$$
$$= \sum_{i=1}^{n}\sum_{k=1}^{m} y_{ki}^2$$
$$= d^2(d_1^{-2} + ... + d_n^{-2})$$
$$\leq d_i^2(d_1^{-2} + ... + d_n^{-2}) \qquad \blacksquare.$$

(b) We prove by induction. For n=1, the statement is obvious. Now let's suppose it's true for n-1, i.e.

For any $\{\gamma_i\}_1^{n-1}$ such that $0 \leq \gamma_i \leq 1$ and $\gamma_1 + ... + \gamma_{n-1} = m$ then there exists an onthogonal matrix

$$\begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n-1,1} \\ a_{12} & a_{22} & \cdots & a_{n-1,2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1,n-1} & a_{2,n-1} & \cdots & a_{n-1,n-1} \end{vmatrix}$$

such that $a_{1i}^2 + ... + a_{mi}^2 = \gamma_i$

For any $\{\beta_i\}_1^n$ such that $0 \leq \beta_i \leq 1$ and $\beta_1 + ... + \beta_n = m$, w.l.o.g suppose that $\beta_1 \geq ... \geq \beta_n$, then $\beta_{n-1} + \beta_n \leq \frac{2m}{n} \leq 1$ and apply the (n-1)th statement with $\gamma_i = \beta_i$ for all i=1,...n-2 and $\gamma_{n-1} = \beta_{n-1} + \beta_n$. The matrix A =

$$\begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n-1,1} \\ a_{12} & a_{22} & \cdots & a_{n-1,2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1,n-1} & a_{2,n-1} & \cdots & a_{n-1,n-1} \end{vmatrix}$$

such that $a_{1i}^2 + ... + a_{mi}^2 = \beta_i$ for i=1,...,n-2 and $a_{1n-1}^2 + ... + a_{mn-1}^2 = \beta_{n-1} + \beta_n$ will be an orthogonal matrix.

Let $Y = \begin{vmatrix} a_{11} & \cdots & a_{n-2,1} & \alpha_1 a_{n-1,1} & -\alpha_2 a_{n-1,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1,n-1} & \cdots & a_{n-2,n-1} & \alpha_1 a_{n-1,n-1} & -\alpha_2 a_{n-1,n-1} \\ 0 & \cdots & 0 & \alpha_2 & \alpha_1 \end{vmatrix}$ where $\alpha_1 = \sqrt{\frac{\beta_{n-1}}{\beta_{n-1}+\beta_n}}$ and $\alpha_2 = \sqrt{\frac{\beta_n}{\beta_{n-1}+\beta_n}}$

We can verify that Y is an orthogonal matrix. Moreover

$$\sum_{k=1}^{m} y_{ki}^2 = \beta_i \text{ for i=1,...,n-2}$$

$$\sum_{k=1}^{m} y_{k,n-1}^2 = \alpha_1^2(\beta_{n-1} + \beta_n) = \beta_{n-1}$$

$$\sum_{k=1}^{m} y_{k,n}^2 = \alpha_2^2(\beta_{n-1} + \beta_n) = \beta_n \qquad \blacksquare.$$

(c) If $m > n/2$ then $n - m < n/2$ and therefore there exists an orthogonal matrix Y such that $y_{m+1,i}^2 + ... + y_{n,i}^2 = 1 - \beta_i$ for i=1,...,n hence $y_{1i}^2 + ... + y_{m1}^2 = \beta_i$.

(d) If $d_i^2(d_1^{-2} + d_2^{-2} + ... + d_n^{-2}) \geq m,$ for i=1,...,n let $d^2 = m(d_1^{-2} + ... + d_n^{-2})_{-1} \leq d_i^2$

With $\beta_i = \frac{d^2}{d_i^2} \leq 1$ and $\beta_1 + .. + \beta_n = m$ we have shown there exists an orthogonal matrix Y such that $y_{1i}^2 + ... + y_{m1}^2 = \beta_i$ for i=1,...,n. A subspace W as defined in (a) exists with an orthonormal basis $\epsilon_1, ..., \epsilon_m$. The full orthonormal basis (of V) is $\epsilon_1, ..., \epsilon_m$. Their coordinates w.r.t. the basis $e_1, ..., e_n$ will be Y^T/d.

8. (a) Let $\chi \in C_0^\infty(I)$ such that $\int \chi(x)dx = 1$ and $C = T_f(\chi)$ which is a constant. For any $\phi \in C_0^\infty(I),$ the function $g(x) = \phi(x) - \chi(x)\int \phi(y)dy$ is also in $C = T_f(\chi)$. Moreover, $\int g(x)dx = 0$. Therefore $h(x) = \int_{-\infty}^{x} g(y)dy$ belongs to $C_0^\infty(I)$ and $h' = g$. So we have

$0 = T_f'(h) = T_f(h') = T_f(g) = T_f(\phi) - C\int \phi(y)dy$ which implies that $T_f = C$ $\blacksquare$

(b) For any $c \in I$ the function $f(x) = \int_c^x g(y)dy$ is continuous on I. Using (a), we only have to prove that $T_f' = T_g$. If $c < x$ then $\forall \phi \in C_0^\infty(I)$:

$$T_f'(\phi) = -T_f(\phi')$$

$$= -\int \phi'(x)f(x)dx$$

$$= -\int \int_c^x g(y)dy\phi'(x)dx$$

$= -\int g(y) \int_{(x,y)\in A} \phi'(x) dx dy$ (using the Fubini theorem where $A = \{(x, y) : c < y < x\}$)

$= \int g(y)\phi(y)dy$

$= T_g(\phi)$ ■