Department of Economics

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S500, First Half.

Homework 4

1. Consider the following experiment: Four fair[1] coins are tossed independently and we only observe the total number of "Heads".

   (a) Describe the sample space of this experiment.

   (b) Characterize the smallest sigma-algebra that contains the following two events separately:

   $$\{\text{ 1 Head showed up}\},\ \{\text{ 2 Heads showed up}\}$$

   From now on, we will denote the resulting sigma-algebra by $\widetilde{\mathcal{F}}$.

   (c) For the sigma-algebra $\widetilde{\mathcal{F}}$, characterize the probability measure $P : \widetilde{\mathcal{F}} \to \mathbb{R}$ that results from the fact that the four coins are fair. [Hint: Because the coins are fair and tossed independently, we have for example that $P\big((H, H, T, T)\big) = 1/2 \cdot 1/2 \cdot 1/2 \cdot 1/2 \cdot = 1/16$. The same argument applies to all possible toss outcomes.]

   (d) Consider the function $X : \Omega \to \mathbb{R}$ that satisfies:

   $$X\big(\text{0 Heads showed up}\big) = 10; \quad X\big(\text{1 Head showed up}\big) = 0; \quad X\big(\text{2 Heads showed up}\big) = -1;$$
   $$X\big(\text{3 Heads showed up}\big) = 5; \quad X\big(\text{4 Heads showed up}\big) = 2.$$

   Is X a random variable on the sigma-field $\widetilde{\mathcal{F}}$? If it is, compute its cdf F_x and compute $E[X]$ using the probability measure P you found in part (c). If X is not a random variable on $\widetilde{\mathcal{F}}$, find the smallest sigma-algebra such that X is a random variable. For this new sigma-algebra, characterize the probability measure P that incorporates the fact that the four coins are fair and compute F_x and $E[X]$ using this probability measure.

   (e) What general features must be shared by all functions $g : \Omega \to \mathbb{R}$ that are measurable on $\widetilde{\mathcal{F}}$?

[1] We say that a coin is fair if it has equal probability (equal to 1/2) of falling "Heads" or "Tails".

2. Show the following

   (a) Suppose F is an algebra defined over some set Ω. Show that $A \in F,\ B \in F \Rightarrow A \cap B \in F$.

   (b) Suppose $F_1,\ F_2$ are algebras defined over some set Ω. Show that $F_1 \cap F_2$ is also an algebra.

   Hint: Use de Morgan's Laws.

3. Let (Ω, F, P) be a probability space. Suppose X is a random variable defined on (Ω, F, P), and that it has a cdf F_x given by

$$F_x(x) = \begin{cases} 0 & \text{if } x \in (-\infty, 0) \\ 1 - e^{-x} & \text{if } x \in [0, 1/4) \\ 1/6 + 1/3 \cdot x & \text{if } x \in [1/4, 1/2) \\ 1/2 & \text{if } x \in [1/2, 3/4) \\ x & \text{if } x \in [3/4, 1) \\ 1 & \text{if } x \in [1, \infty) \end{cases}$$

   Compute $E[X]$, $\mathrm{Var}[X]$ and $m(t) = E\left[e^{tX}\right]$ for a fixed $t > 0$. Is $m(t)$ differentiable at $t = 0$? If it is, compute $m'(0)$.

4. We call the smallest sigma-algebra that contains a subset M of X the sigma-algebra generated by M.

   (a) In the definition of the Borel sigma-algebra, if we replace the open sets by the closed ones will we get the same sigma-algebra ?

   (b) Let M be a subset of R. What is sigma-algebra generated by M ?

   (c) Let $A = \{(p,q) : p,q \in Q\}$. Prove that sigma-algebra generated by A is the Borel sigma-algebra on R.

5. A measure μ is function defined on a sigma-algebra $\mathcal{B}$ over a set X such that:

   * $\mu(\emptyset) = 0$,

   * (countable additivity) if (A_i) are disjoint pairwise then $\mu\left(\bigcup_i A_i\right) = \sum_i \mu(A_i)$. The triple $(X, \mathcal{B}, \mu)$ is called the measure set, and the elements of $\mathcal{B}$ are μ-measurable. Prove that for any sequence $(A_i), i \in N$ in $\mathcal{B}$:

   (a) If (A_i) is increasing in $\mathcal{B}(i.e. A_i \subseteq A_j$ for any $i < j$ and $i, j \in N)$ then $\mu\left(\bigcup_i A_i\right) = \sup_i \mu(A_i) = \lim_i \mu(A_i)$

   (b) If (A_i) is decreasing (i.e. $A_i \supseteq A_j$ for any $i < j$) and there exists k such that $\mu(A_k) < \infty$ then

   $$\mu\left(\bigcap_i A_i\right) = \inf_i \mu(A_i) = \lim_i \mu(A_i)$$

   (c) Give an example where (b) is wrong if all $\mu(A_i)$ are infinite.