Department of Economics

Princeton University

S500, First Half.

Suggested Answers to Homework 4

1. (a) We have to characterize carefully the sample space. Naturally, there are only five possible outcomes: "0 Heads", "1 Head", "2 Heads", "3 Heads", "4 Heads". Each one of these outcomes is linked to the physical outcome of tossing four coins as follows:

   "0 Heads": $\big\{(T,T,T,T)\big\}$

   "1 Head": $\big\{(H,T,T,T),(T,H,T,T),(T,T,H,T),(T,T,T,H)\big\}$

   "2 Heads": $\big\{(H,H,T,T),(H,T,H,T),(H,T,T,H),(T,H,H,T),(T,H,T,H),(T,T,H,H)\big\}$

   "3 Heads": $\big\{(H,H,H,T),(H,H,T,H),(T,H,H,H),(H,T,H,H)\big\}$

   "4 Heads": $\big\{(H,H,H,H)\big\}$

   So "0 Heads" consists of one coin-tossing outcome, "1 Head" consists of 4 such outcomes, "2 Heads" consists of 6, "3 Heads" consists of 4" and "4 Heads" consists of 1. These exhaust all possible coin-tossing outcomes: 16 in total. From now on, we will abbreviate the sample space as $\Omega = \{0,1,2,3,4\}$, keeping in mind what is behind each possible outcome.

   (b) We start with $\big\{\{1\},\{2\}\big\}$. Next, we add $\emptyset$, Ω, next we add the union of the elements that are in there already: we only need to add $\{1,2\}$. Next, we add the complement of each element: $\{1\}^c = \{0,2,3,4\}$, $\{2\}^c = \{0,1,3,4\}$, $\{1,2\}^c = \{0,3,4\}$. So far we have $\big\{\emptyset,\Omega,\{1\},\{2\},\{1,2\},\{0,2,3,4\},\{0,1,3,4\},\{0,3,4\}\big\}$. We can verify that this set contains the unions of all the elements in the set. Therefore, we stop here. This is the smallest sigma-algebra that contains $\{1\}$, $\{2\}$:

$$\widetilde{\mathcal{F}} = \big\{\emptyset,\Omega,\{1\},\{2\},\{1,2\},\{0,2,3,4\},\{0,1,3,4\},\{0,3,4\}\big\}$$

(c) We have

$$\{\omega \in \Omega : X \leq x\} = \begin{cases} \emptyset & \text{if } x \in (-\infty, -1) \\ \{2\} & \text{if } x \in [-1, 0) \\ \{1,2\} & \text{if } x \in [0, 2) \\ \{1,2,4\} & \text{if } x \in [2, 5) \\ \{1,2,3,4\} & \text{if } x \in [5, 10) \\ \Omega & \text{if } x \in [10, \infty) \end{cases}$$

$\{1,2,4\} \notin \widetilde{\mathcal{F}}$ and $\{1,2,3,4\} \notin \widetilde{\mathcal{F}}$. Therefore X is not a random variable (i.e, it's not measurable) in this sigma-algebra. The smallest sigma-algebra such that X is a random variable is the one generated by $\{\{2\}, \{1,2\}, \{1,2,4\}, \{1,2,3,4\}\}$.

There are $2^4 = 16$ possible outcomes of tossing four coins. If each coin is fair, the probability of observing each outcome is $(1/2) \cdot (1/2) \cdot (1/2) \cdot (1/2) = 1/16$. A probability measure that is induced by the fact that the four coins are fair is the one that satisfies the Kolmogorov Axioms of Probability, and assigns the following probabilities to the five possible outcomes of the experiment:

$$P(\text{“0 Heads”}) = \Pr\Big(\{(T,T,T,T)\}\Big) = 1/16$$

$$P(\text{“1 Head”}) = \Pr\Big(\{(H,T,T,T),(T,H,T,T),(T,T,H,T),(T,T,T,H)\}\Big) = 4 \cdot (1/16) = 1/4$$

$$\begin{aligned} P(\text{“2 Heads”}) &= \Pr\Big(\{(H,H,T,T),(H,T,H,T),(H,T,T,H),(T,H,H,T),(T,H,T,H),(T,T,H,H)\}\Big) \\ &= 6 \cdot (1/16) = 3/8 \end{aligned}$$

$$P(\text{“3 Heads”}) = \Pr\Big(\{(H,H,H,T),(H,H,T,H),(T,H,H,H),(H,T,H,H)\}\Big) = 4 \cdot (1/16) = 1/4$$

$$P(\text{“4 Heads”}) = \Pr\Big(\{(H,H,H,H)\}\Big) = 1/16$$

The cdf F_x is given by:

$$F_x(x) = \Pr\{\omega \in \Omega : X \leq x\} = \begin{cases} \Pr(\emptyset) = 0 & \text{if } x \in (-\infty, -1) \\ \Pr(\{2\}) = 3/8 & \text{if } x \in [-1, 0) \\ \Pr(\{1,2\}) = 1/4 + 3/8 = 5/8 & \text{if } x \in [0, 2) \\ \Pr(\{1,2,4\}) = 1/4 + 3/8 + 1/16 = 11/16 & \text{if } x \in [2, 5) \\ \Pr(\{1,2,3,4\}) = 1/4 + 3/8 + 1/16 + 1/4 = 15/16 & \text{if } x \in [5, 10) \\ \Pr(\Omega) = 1 & \text{if } x \in [10, \infty) \end{cases}$$

The function F_x satisfies all the requirements of a well-defined cdf: $\lim_{x\to\infty} F_x(x) = 1$, $\lim_{x\to-\infty} F_x(x) = 0$, $\int_{-\infty}^{\infty} dF(x) = 1$ and F_x is right-hand continuous. The resulting cdf

F_x is a step-function, we know X is a discrete random variable, and

$$E[X] = \int_{-\infty}^{\infty} x dF(x) = X(0) \cdot P(0) + X(1) \cdot P(1) + X(2) \cdot P(2) + X(3) \cdot P(3) + X(4) \cdot P(4)$$
$$= \frac{10}{16} - \frac{3}{8} + \frac{5}{4} + \frac{2}{16} = \frac{13}{8}$$

(d) Taking a look at $\widetilde{\mathcal{F}}$, we observe that the outcomes "0 Heads", "3 Heads" and "4 Heads" are always together. Therefore, any measurable function $g : \Omega \to \mathbb{R}$ on this sigma-algebra must satisfy $g(0) = g(3) = g(4)$. The values $g(1)$ and $g(2)$ are unrestricted.

2. (a) If $A, B \in F$ and F is an algebra, then $A^c \in F$ and $B^c \in F$. Then $A^c \cup B^c \in F$ and therefore $\left(A^c \cup B^c\right)^c \in F$. By de Morgan's Laws, $A \cap B = \left(A^c \cup B^c\right)^c$. Therefore $A \cap B \in F$.

   (b) First, note that $\Omega \in F_1$ and $\Omega \in F_2$. Therefore $\Omega \in F_1 \cap F_2$. Next: if $A \in F_1 \cap F_2$, then $A^c \in F_1$ and $A^c \in F_2$. Therefore, if $A \in F_1 \cap F_2$, then $A^c \in F_1 \cap F_2$. Next, suppose A and B are in $F_1 \cap F_2$. Then $A, B \in F_1$ and $A, B \in F_2$. Therefore, $A \cup B \in F_1$ and $A \cup B \in F_2$. Consequently $A \cup B \in F_1 \cap F_2$ and we're done.

3. The cdf Fx has three jumps: a jump of magnitude $1/4 - (1 - e^{-1/4})$ when $x = 1/4$, a jump of magnitude $1/6$ when $x = 1/2$ and a final jump of magnitude $1/4$ when $x = 3/4$, it is continuous at all other points, and differentiable at all other points except at $x = 1$, where it is also continuous. It is constant at the value $1/2$ in the interval $x \in [1/2, 3/4)$. The function is continuous from the right, satisfies $\lim_{x \to -\infty} F_x(x) = 0$ and $\lim_{x \to \infty} F_x(x) = 1$. Finally:

$$\int_{-\infty}^{\infty} dF(x) = 0 + \int_0^{1/4} e^{-x} dx + \frac{1}{4} - \left[1 - e^{-1/4}\right] + \frac{1}{3} \int_{1/4}^{1/2} 1 \cdot dx + \frac{1}{6} + \frac{1}{4} + \int_{3/4}^{1} 1 \cdot dx + 0 = 1 \checkmark$$

We have

$$E[X] = \int_0^{1/4} x e^{-x} dx + \frac{1}{4}\left[\frac{1}{4} - \left(1 - e^{-1/4}\right)\right] + \frac{1}{3} \int_{1/4}^{1/2} x dx + \frac{1}{6} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{3}{4} + \int_{3/4}^{1} x dx$$
$$= \frac{4}{3} - e^{-1/4}$$
$$E[X^2] = \int_0^{1/4} x^2 e^{-x} dx + \left(\frac{1}{4}\right)^2 \left[\frac{1}{4} - \left(1 - e^{-1/4}\right)\right] + \frac{1}{3} \int_{1/4}^{1/2} x^2 dx + \frac{1}{6} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 + \int_{3/4}^{1} x^2 dx$$
$$= \frac{337}{144} - \frac{5}{2} e^{-1/4}$$

and so $\mathrm{Var}[X] = E[X^2] - \Big(E[X]\Big)^2 \approx 0.0857$. Next:

$$m(t) = E[e^{tX}] = \int_0^{1/4} e^{tx} \cdot e^{-x} dx + e^{t/4}\left[\frac{1}{4} - \left(1 - e^{-1/4}\right)\right] + \frac{1}{3} \int_{1/4}^{1/2} e^{tx} dx + \frac{1}{6} \cdot e^{t/2} + \frac{1}{4} \cdot e^{3t/4} + \int_{3/4}^{1} e^{tx} dx$$
$$= \frac{1}{1-t} + \frac{e^t}{t} + \frac{e^{3t/4}(t-4)}{4t} + \frac{e^{(t-1)/4} t}{t-1} + \frac{e^{t/2}(2+t)}{6t} - \frac{e^{t/4}(4+9t)}{12t}$$

This function $m(t)$ is not defined at $t = 0$. Therefore it's not differentiable there. This is the end of the problem. However, let's find out a little more: we can verify that

$$\lim_{t \to 0} m(t) = 1$$

Also, we can see that $m'(t)$ exists for all $t \neq 0$. It turns out that:

$$\lim_{t \to 0} \ m'(t) = \frac{4}{3} - e^{-1/4} = E[X]$$

$m''(t)$ also exists for all $t \neq 0$. It turns out that:

$$\lim_{t \to 0} \ m''(t) = \frac{337}{144} - \frac{5}{2}e^{-1/4} = E[X^2]$$

Spooky? Not after you learn the concept of a moment generating function, which –if it hasn't happened yet– will happen in your first Ph.D course in Econometrics.

4. (a) For any $a, b \in R$, take 2 sequences $\{a_n\}$ and $\{b_n\}$ such that $a_n \to a$ and $b_n \to b$. We have

$[a, b] = R \setminus \bigcup_n [(-\infty, a_n) \bigcup (b_n, \infty)]$ with $a_n < a < b < b_n$ which implies that the sigma-algebra generated by the open sets (the Borel sigma-algebra as defined in the notes) contains the one generated by the closed sets.

$(a, b) = \bigcup_n (a_n, b_n)$ with $a_n, b_n \in (a, b)$ which implies that the sigma-algebra generated by the closed sets contains the Borel sigma-algebra.

Therefore the two definitions are identical.

(b) Consider the intersection $\mathcal{B}_0$ of all the sigma-algebras $\mathcal{B}_i$ that contain M. For any element A of $\mathcal{B}_0$, A belongs to all $\mathcal{B}_i$, therefore A^C belongs to all $\mathcal{B}_i$, so $A^C \in \mathcal{B}_0$. Also, if (A_i) belong to $\mathcal{B}_0$ then (A_i) belong to all $\mathcal{B}-)$. Their countable union $\bigcup_i (A_i)$ also belong to all $\mathcal{B}_i$, therefore is an element of $\mathcal{B}_0$. We can conclude that $\mathcal{B}_0$ is a sigma-algebra. Obviously it is the smallest one that contains M.

(c) Call M the subset of all the open interval of R. We have $A \subset M$. So the Borel sigma-algebra contains the sigma-algebra generated by A. However, for any $a \in R$, there exists $(p_n) \in Q$ such that $p_n \to a$. Therefore $(a, b) = \bigcup_n (p_n, q_n)$ where $p_n \to a, q_n \to b$ and $p_n, q_n \in (a, b)$. Therefore the sigma-algebra generated by A contains the Borel sigma-algebra $\blacksquare$.

5. (a) Let $B_1 = A_1$ and $B_i = A_i \setminus A_{i-1}$ for any $i > 1$. Obviously the B_i are pairwise disjoint, so

$$\begin{aligned} \mu\left(\bigcup_i A_i\right) &= \mu\left(\bigcup_i B_i\right) \\ &= \inf_i \mu(A_i) = \lim_i \mu(A_i) \qquad \blacksquare \end{aligned}$$

(b) Let $B_j = A_k \setminus A_j$ for any $j \geq k . B_j$ is an increasing sequence in $\mathcal{B}$ whose union is $A_k \setminus A$ where $A = \bigcap_i A_i$, we then can apply (a) which gives:

$$\begin{aligned} \mu(A_k) - \mu(A) &= \mu\left(\bigcup_i B_i\right) \\ &= \lim_i \mu(B_i) \\ &= \lim_i \mu(A_k \setminus A_i) \\ &= \mu(A_k) - \lim_i (A_i) \qquad \blacksquare \end{aligned}$$

(c) Let X=N, $\mu(A) = card(A)$ and $A_i = \{1, ..., p\} \bigcup \{i, i+1, ...\}$ for any $p \in N$. Then $\mu\left(\bigcap_i A_i\right) = \mu(\{1, ...p\}) = p$ while $\lim_i \mu(A_i) = \infty$.