Equivalence of 2D color codes (without translational symmetry) to surface codes

Arjun Bhagoji\textsuperscript{1} Pradeep Sarvepalli\textsuperscript{1}

\textsuperscript{1}Department of Electrical Engineering
Indian Institute of Technology, Madras

IEEE International Symposium on Information Theory 2015, Hong Kong
Motivation

- Surface codes, a class of topological codes, have many advantages (local stabilizer generators, low complexity decoders etc.)

- Limited in terms of transversal gates that can be implemented

- Color codes, another class of topological codes, can implement entire Clifford group transversally

So, they are inequivalent? **No!**

Bombin, Duclos-Cianci and Poulin (2012) showed that translationally invariant 2D color codes can be mapped to a finite number of copies of Kitaev’s toric code.

**Question:** What about translation variant codes?
Surface codes

- Topological codes are a class of quantum codes where information is stored in topological degrees of freedom.
- Surface codes are topological codes with qubits attached to the edges of a lattice embedded in a given closed surface.
- Each face (plaquette) and vertex of the lattice defines a stabilizer acting on the surrounding qubits.

**Figure:** Left: Stabilizers on a copy of Kitaev’s toric code. Right: $E_1$ is a homologically trivial error, can be detected and corrected. $E_2$ is homologically non-trivial, cannot be detected, interacts with the encoded qubits.
Color codes

- Different class of topological codes, embedded on 2-colexes (trivalent, 3-face-colorable complexes)
- Qubits attached to every vertex
- Each face of the lattice defines a stabilizer acting on the qubits in its boundary

Figure: Hexagonal color code
Charges and syndromes

- Think of each violated check (stabilizer anti-commuting with error) as a charge

- 4 types of charges on a surface code: i) electric ($\epsilon$) on vertices, ii) magnetic ($\mu$) on the plaquettes, iii) composite electric and magnetic charge ($\epsilon\mu$) on both plaquettes and vertices, and iv) the vacuum ($\iota$)

- Only two of these are independent, which we choose to be $\epsilon$ (due to $Z$-type) and $\mu$ (due to $X$-type)

- Color code charges also have color associated with them.

- 16 types of charges on a color code (Bombin et.al., 2012) with 2 independent pairs
Hopping operators

- Any element of the Pauli group that moves charges
- Stabilizers can be viewed as a combination of hopping operators moving charges back to their initial position

Figure: Hopping operators on the surface code (left) and color code (right)
Mapping between color codes and surface codes

- **Observation**:  
  1. 4 types of charges on a surface code and 16 types of charges on a color code.  
  2. 2 pairs of independent charges on the color code

⇒ Decompose the color code into 2 surface codes, mapping each charge pair from the color code to one of the surface codes

- Electric charges on the surface codes live on the vertices while the magnetic charges live on the plaquettes.
- Consider the charges on the color code $\Gamma$: all live on plaquettes

**How do we relate the two?**

**Notation**: For what follows, we denote the faces, edges and vertices of the embedding of a graph $\Gamma$ by $F(\Gamma)$, $E(\Gamma)$ and $V(\Gamma)$.  

$$B_f^\sigma = \prod_{v \in f} \sigma_v$$  

denotes the stabilizers on the color code, where $\sigma \in X, Z$. 
Desirable constraints on map

The map $\pi$ between the color code and the surface codes should have the following desirable properties:

1. Linear
2. Invertible (i.e. map should be bijective)
3. Local
4. Efficiently computable
5. Preserve the commutation relations between the Pauli error operators on $V(\Gamma)$, i.e. $\mathcal{P}_{V(\Gamma)}$
6. Consistent in the description of the movement of charges on the color code and surface codes.

Focus will be on the consistency of charge movement and the preservation of commutation relations. Time permitting, other constraints will be discussed.
Contract all the $c$-colored plaquettes in the embedding of $\Gamma$. This will give rise to a new graph $\tau_c(\Gamma)$.

**Figure:** Illustrating the contraction of a color code via $\tau_c$ and the resultant surface code. Only portions of the codes are shown.
Each $c$-colored face in $\Gamma$ can host $\epsilon_c$ and $\mu_c$.

Both these charges live on the vertices of a contracted lattice.

Identify $\epsilon_c \equiv \epsilon_1$ and $\mu_c \equiv \epsilon_2$.

Now choose the magnetic charges on $\Gamma_1$ and $\Gamma_2$ from $\epsilon_{c'}$, $\epsilon_{c''}$ and $\mu_{c'}$, $\mu_{c''}$. How?

**Lemma (Charge mapping)**

Let $c, c', c''$ be three distinct colors. Then, $\{\epsilon_c, \mu_c\}$ and $\{\epsilon_{c'}, \mu_{c'}\}$ are permissible pairings of the charges so that the color code on $\Gamma$ can be mapped to a pair of surface codes on $\Gamma_i = \tau_c(\Gamma)$. In other words, $\epsilon_1 \equiv \epsilon_c$, $\mu_1 \equiv \mu_{c'}$, $\epsilon_2 \equiv \mu_c$ and $\mu_2 \equiv \epsilon_{c'}$, where $\epsilon_i$ and $\mu_i$ are the electric and magnetic charges of the surface code on $\Gamma_i$. 
Figure: Mapping the electric charge hopping operators from the color code to the surface codes
Mapping elementary hopping operators

Lemma (Elementary hopping operators)

Let $f, f' \in F_c(\Gamma)$ where the edge $(u, v)$ is incident on $f$ and $f'$. Then, the following choices reflect the charge movement on $\Gamma$ onto the surface codes on $\Gamma_i$.

\[
\pi(H^{e_c}_{u,v}) = \begin{bmatrix} Z_{\tau(u)} \end{bmatrix}_1 = \begin{bmatrix} Z_{\tau(v)} \end{bmatrix}_1
\]

(1)

\[
\pi(H^{\mu_c}_{u,v}) = \begin{bmatrix} Z_{\tau(u)} \end{bmatrix}_2 = \begin{bmatrix} Z_{\tau(v)} \end{bmatrix}_2
\]

(2)

where $[T]_i$ indicates the instance of the surface code on which $T$ acts.

Now if $f, f' \in F_{c'}(\Gamma)$ and $(u, v) \in E_{c'}(\Gamma)$ such that $u \in f$ and $v \in f'$ and $H^{e_{c'}}_{u,v}$ and $H^{\mu_{c'}}_{u,v}$ are chosen to be independent hopping operators of $f$, then

\[
\pi(H^{e_{c'}}_{u,v}) = \begin{bmatrix} X_{\tau(u)}X_{\tau(v)} \end{bmatrix}_2; \pi(H^{\mu_{c'}}_{u,v}) = \begin{bmatrix} X_{\tau(u)}X_{\tau(v)} \end{bmatrix}_1.
\]

(3)
Dependent hopping operators

Not all the hopping operators on a particular face are independent.

Lemma (Dependent hopping operators)

Let $f \in F_{c''}(\Gamma)$ and $1, \ldots, 2\ell_f$ be the vertices in its boundary so that $(2i - 1, 2i) \in E_c(\Gamma)$, $(2i, 2i + 1) \in E_{c'}(\Gamma)$ for $1 \leq i \leq \ell_f$ and $2\ell_f + 1 \equiv 1$. If $\pi$ is invertible, then $\pi(B_f^\sigma) \neq I$ and there are $4\ell_f - 2$ independent elementary hopping operators along the edges of $f$. 
Mapping the single qubit errors

Let the operators $H_{1,2\ell_f}^{c'_{\ell_f}}$ and $H_{2m,2m+1}^{c'_{2m}}$ be the dependent ones.

$$
\begin{bmatrix}
Z_1Z_2 \\
Z_3Z_4 \\
\vdots \\
Z_{2\ell-1}Z_{2\ell}
\end{bmatrix} \xrightarrow{\pi} \begin{bmatrix}
Z \\
Z \\
\vdots \\
Z_{1\cup2}
\end{bmatrix}
$$

$$
\begin{bmatrix}
Z_2Z_3 \\
Z_4Z_5 \\
\vdots \\
Z_{2\ell-2}Z_{2\ell-1}
\end{bmatrix} \xrightarrow{\pi} \begin{bmatrix}
X \\
X \\
\vdots \\
X \\
X
\end{bmatrix}
$$
Single qubit error splitting

\[
\begin{bmatrix}
X_2X_3 \\
\vdots \\
X_{2m-2}X_{2m-1} \\
X_{2m+2}X_{2m+3} \\
\vdots \\
X_{2\ell}X_1
\end{bmatrix} \xrightarrow{\pi} \begin{bmatrix}
X \\
\vdots \\
X \\
X \\
\vdots \\
X
\end{bmatrix} \xrightarrow{\pi} \begin{bmatrix}
X & X \\
\vdots & \ddots & X & X \\
X & X & \ddots & \ddots \\
X
\end{bmatrix}
\]

(6)

Lemma (Splitting)

The following choices lead to an invertible \( \pi \) while respecting the commutation relations with the independent hopping operators.

\[
\pi(gX_1) = [X_{\tau(1)}]_1 \text{ where } g \in \{ I, B_f^X, B_f^Y, B_f^Z \} \tag{7}
\]

\[
\pi(gZ_{2m}) = [X_{\tau(2m)}]_2 \text{ where } g \in \{ I, B_f^X \} \tag{8}
\]
Any 2D color code (on a 2-colex \( \Gamma \) without parallel edges) is equivalent to a pair of surface codes \( \tau(\Gamma) \) under the map \( \pi \).

- Mapping can be done locally for each \( c'' \)-colored face
- Stabilizers on the color code are mapped to stabilizers on the surface code, so code capabilities are preserved
- Decoding of color codes is a possible application
- Mapped errors can be decoded on the surface code, and uniquely lifted back to the color code, as the map is bijective
Map for $X$-type errors on the hexagonal color code
Map for $X$-type errors on the hexagonal color code
### Comparison with previous work

<table>
<thead>
<tr>
<th>Authors</th>
<th>Translation invariance reqd.</th>
<th>No. of copies</th>
<th>Ancilla qubits reqd.</th>
<th>Injective</th>
<th>Conseq. for decoding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bombin et. al.</td>
<td>Yes</td>
<td>Finite, could be many</td>
<td>Sometimes</td>
<td>Yes</td>
<td>High complexity</td>
</tr>
<tr>
<td>Delfosse et. al.</td>
<td>No</td>
<td>3</td>
<td>Not sure</td>
<td>Yes</td>
<td>No pre-image for some errors</td>
</tr>
<tr>
<td>Present work</td>
<td>No</td>
<td>2</td>
<td>No</td>
<td>Yes</td>
<td>Non-CSS nature, up to 2 decoders per surface code</td>
</tr>
</tbody>
</table>

Bhagoji, Sarvepalli  | Equivalence of 2D color codes (without translational symmetry) to surface codes | ISIT 2015 | 19 / 20
That’s all folks!