MAE 545: Lecture 19 (12/2)

Wrinkled surfaces
Compression of stiff thin membranes on liquid and soft elastic substrates

stiff membrane

Liquid substrate

Elastic substrate $E_s \ll E_m$

 Compression

10 $\mu$m thin sheet of polyester on water $\lambda_0 \sim 1.6\text{cm}$

$\sim$10 $\mu$m thin PDMS (stiffer) sheet on PDMS (softer) substrate $\lambda_0 \sim 70\mu$m

Compression of stiff thin membranes on liquid substrates

\[ U_c \sim A \times E_m d \times \varepsilon^2 \]

\[ U_b, U_p \sim A\varepsilon \sqrt{E_m d^3 \rho g} \]

wrinkles are stable above the critical strain \( \varepsilon > \varepsilon_c \sim \sqrt{\frac{\rho g d}{E_m}} \)

wavelength of wrinkles \( \lambda \sim \left( \frac{E_m d^3}{\rho g} \right)^{1/4} \)

amplitude of wrinkles at the critical strain \( h_0^* \sim \lambda \sqrt{\varepsilon_c} \sim d \)
Compression of stiff thin membranes on liquid substrates

Scaling analysis

\[ \lambda \sim \left( \frac{E_m d^3}{\rho g} \right)^{1/4} \]

Exact result

\[ \lambda = 2\pi \left( \frac{\kappa}{\rho g} \right)^{1/4} \]

\[ \ln(\lambda) \]

\[ \ln(\kappa/\rho g) \]
Compression of stiff thin membranes on liquid substrates

How to go beyond the simple scaling analysis to determine the nonlinear post-buckling behavior?

Find shape profile $h(s)$ that minimizes total energy

$$U_b + U_p = W \int_0^L \frac{ds}{2} \left[ \frac{\kappa h''^2}{(1 + h'^2)^3} + \rho gh^2 \sqrt{1 - h'^2} \right]$$

subject to constraint

$$L - \Delta = \int_0^L ds \sqrt{1 - h'^2}$$

Compression of stiff thin membranes on liquid substrates

Comparison between theory (infinite membrane) and experiment

\( h/\lambda \)

\( \epsilon = 0.15 \)

\( \epsilon = 0.30 \)

\( \epsilon = 0.80 \)

\( s/\lambda \)

\( A_0/\lambda \)

\( A_1/\lambda \)


Compression of stiff thin membranes on soft elastic substrates

Compression energy of thin membrane

\[ U_c \sim A \times E_m d \times \epsilon^2 \]

membrane area

\[ A = WL \]

membrane 3D Young’s modulus

\[ E_m \]

strain

\[ \epsilon = \frac{\Delta}{L} \]

substrate 3D Young’s modulus

\[ E_s \]
Compression of stiff thin membranes on soft elastic substrates

**assumed profile**

\[ h(s) = h_0 \cos\left(\frac{2\pi s}{\lambda}\right) \]

\[ h_0 = \frac{\lambda}{\pi} \sqrt{\frac{\Delta}{L}} = \frac{\lambda \sqrt{\epsilon}}{\pi} \]

**amplitude of wrinkles**

**deformation of the soft substrate decays exponentially away from the surface**

\[ h(s, y) \approx h_0 \cos\left(\frac{2\pi s}{\lambda}\right)e^{-y/\lambda} \]

**bending energy of stiff membrane**

\[ U_b \approx A \times \kappa \times \frac{1}{R^2} \sim A \times E_m d^3 \times \frac{h_0^2}{\lambda^4} \sim \frac{A E_m d^3 \epsilon}{\lambda^2} \]

**deformation energy of soft substrate**

\[ U_s \sim V \times E_s \times \epsilon_s^2 \sim A \lambda \times E_s \times \frac{h_0^2}{\lambda^4} \sim A E_s \lambda \epsilon \]

**minimize total energy** \((U_b+U_s)\) **with respect to** \(\lambda\)

\[ \lambda \approx d \left(\frac{E_m}{E_s}\right)^{1/3} \]

\[ U_b, U_s \sim A d \epsilon \left(\frac{E_s^2 E_m}{E_m}\right)^{1/3} \]
Compression of stiff thin membranes on soft elastic substrates

$U_c \sim A \times E_m d \times \epsilon^2$

$U_b, U_s \sim Ad \epsilon \left( \frac{E_s^2 E_m}{3} \right)^{1/3}$

wrinkles are stable for large strains

$\epsilon > \epsilon_c \sim \left( \frac{E_s}{E_m} \right)^{2/3}$

wavelength of wrinkles

$\lambda \sim d \left( \frac{E_m}{E_s} \right)^{1/3}$

amplitude of wrinkles at the critical strain

$h_0^* \sim \lambda \sqrt{\epsilon_c} \sim d$
Compression of stiff thin membranes on liquid and soft elastic substrates

wavelength of wrinkles on liquid substrates

\[ \lambda = 2\pi \left( \frac{k}{\rho g} \right)^{1/4} \]

wavelength of wrinkles on soft elastic substrates

\[ \lambda = 2\pi \left( \frac{3k}{E_s} \right)^{1/3} \]
Compression of stiff thin membranes on soft elastic substrates

In order to explain period doubling (quadrupling, ...) one has to take into account the full nonlinear strain tensor of the soft substrate

\[2u_{ij}^s = \left( \partial_i u_j^s + \partial_j u_i^s \right) + \sum_k \partial_i u_k^s \partial_j u_k^s\]

\[i, j, k \in x, y, z\]
Uniform compression of stiff thin membranes on soft elastic substrates

As the applied overstress increases beyond the formation of the hexagonal mode, a transition point to more...

The preference of our experimental system to assume the hexagonal mode at low overstress is overwhelming, despite...

The experimental observations noted here have motivated us to look for a theoretical explanation of why the square...

Lin and Yang, 2007

Huang et al., 2005;...

The herringbone pattern is dominant with occasional defects. An exceptionally well-organized herringbone pattern is seen in hexagonal lattice of the original pattern, perhaps due to kinetic considerations of forming an entirely new pattern with energetically favorable herringbone patterns is observed. However, experimentally there is a tendency to maintain the above that are otherwise inexplicable when the films are taken to be flat.

Progression of modes observed experimentally for the UVO-treated PDMS system with increasing overstress as described in the text. The UVO treatment times from left to right are 10, 15, 20, 30, 45, and 60 min. Starting from a pure hexagonal pattern, slight increases in the overstress cause isolated hexagons to buckle into the substrate (e.g., substrate system is flat. Furthermore, we have only observed hexagonal patterns for which the regions inside the hexagons coalesce into grooves link to form even longer grooves, but in general they tend to remain the product of just 2–3 hexagons. The overall pattern formed by regular polygons are the equilateral triangle mode and square mode. The mechanism by which the hexagonal mode transitions to a more energy-minimizing pattern is seen in analog to the labyrinth pattern reported for homogeneously initiated wrinkling at high overstress (Huang et al., 2005; Lin and Yang, 2007). In contrast, well-ordered herringbone patterns develop for systems that bypass the lower energy checkerboard modes, the square mode has the lowest energy. (iii) The hexagonal mode and triangular mode have identical energy in the buckled state, to the order obtained here, and a continuous transition exists from one to the other at constant compression. (v) A slight curvature of the film is likely to be playing a critical role in the mode selection observed experimentally and in determining the sign of the hexagonal deflection.

Embedded within the paper are several auxiliary findings: (i) The only modes whose nodal lines coincide with a substrate. (ii) The only mode whose nodal lines coincide with a Checkerboard mode, the square mode has the lowest energy. (iii) The hexagonal mode and triangular mode have identical energy in the buckled state, to the order obtained here, and a continuous transition exists from one to the other at constant compression. (v) A slight curvature of the film is likely to be playing a critical role in the mode selection observed experimentally and in determining the sign of the hexagonal deflection.

All protocols produce equivalent results!
Compression of stiff thin membranes on a spherical soft substrates

Spherical shells are compressed by reducing internal pressure

\[ R = 20 \text{mm} \]

hexagonal phase  bistable phase  labyrinth phase

Phase diagram

\[ \frac{\Delta p}{p_c} - 1 = \frac{\epsilon}{\epsilon_c} - 1 \]

Compression of stiff thin membranes on a spherical soft substrates

Swelling of gels

Phase diagram

\[ \Delta p/p_c - 1 = \varepsilon/\varepsilon_c - 1 \]

Effective radius

Unwrinkled

Hysteresis

Hexagonal phase

Bistable phase

Labyrinth phase

Modifying radius \( R \)

Modifying membrane thickness \( d \)

\[ R = 381 \mu m \]

\[ R = 522 \mu m \]

How are villi formed in guts?

Villi increase internal surface area of intestine for faster absorption of digested nutrients.
Lumen patterns in chick embryo

- DAPI marks cell nuclei
- aSMA marks smooth muscle actin
- EX: age of chick embryo in days

Stiff muscles grow slower than softer mesenchyme and endoderm layers

- E8
- E10
- E12

- radial compression due to differential growth produces striped wrinkles

- endoderm mesenchymal muscle

- 100 µm

A. Shyer et al., Science 342, 212 (2013)
elaborations of the lining of the gut, essential for providing compressive stresses that lead to their buckling and folding. Layers of the gut, which restrict the expansion of the face of the gut transforms from a smooth surface to a convoluted morphology. In humans, as well as in other species, longitudinal muscle differentiates just anterior to the circular layer, coincident with the formation of villi.

Formation of longitudinal muscles at E13 produces longitudinal compression. Last, at E16, a saturated longitudinal layer of muscle differentiates just anterior to the circular layer (arrowhead), coincident with the formation of villi. The villi of the human and chick gut are formed in similar stepwise progressions, wherein the mesenchyme and the growing endoderm and muscle layers interact. The mesenchyme, generating the outer endodermally derived epithelium and outer muscle layers, enters into ridges, zigzags, and honeycombs. These processes are described in detail in the images of corresponding gut lumen pattern; longitudinal axis runs top to bottom.

Lumen patterns in chick correspond with differential differentiation of the distinct smooth muscle unit notes, the primitive interior to the circular layer (arrowhead), coincident with the formation of villi. (Right) Whole-mount stained for nuclei [4',6-diamidino-2-phenylindole (DAPI)] during development. (Left) Schematic illustrating the process of gut patterning over time.

A quantitative computational analysis of muscle layer formation in the chick reveals a linear relationship between the number of muscle layers and the number of villi produced. This finding supports the hypothesis that villification is a process that depends on the sequential differentiation of muscle layers.

These results have implications for understanding the developmental mechanisms underlying gut morphogenesis and provide insights into the potential therapeutic strategies for the treatment of intestinal diseases.
Lumen patterns in chick embryo

Villi start forming at E16 because of the faster growth in valleys

The same mechanism for villi formation also works in other organisms!

A. Shyer et al., Science 342, 212 (2013)
Why are guts shaped like that?
**Guts in chick embryo**

Surgically removed guts from chick embryo

![Images of gut development at different stages](E5, E8, E12, E16)

**Tube** straightens after separation from mesentery

**Tube** grows faster than mesentery sheet!

- Tube
- Mesentery

**Geometric and mechanical measurements of chick gut.**

- Tube cross-section shown in inset.
- Figure 3
- Scaling laws for loop period, radius and number of loops (mimicking the mesentery; see Supplementary Information).
- Parameters involved in the physical model.

\[ \sigma (\text{Pa}) \]

\[ L \]

\[ d \]

**Morphology of loops in the chick gut.**

- a)
- b)
- c)
- d)
- e)

- Surgical removal of guts from chick embryos
- Cranial (superior) mesenteric artery (SMA), at day E4, before loops formed.
- Varying the moduli, strain, growth strain, physiological strains, we use the linearization shown by the black lines, to approximate the behavior of the system.

\[ C_1, C_2, C_3 \]

With mesentery shrunk while the gut tube straightens almost completely.

- Differences in proliferation rates at different cross-sections (Supplementary Fig. 2).
- Each blue bar represents the average number of phospho-H3-positive cells per mm length measurement of one isolated segment.

**Progeny of the gut loops.**

- Cranial (superior) mesenteric artery has been cut out (but not the mesentery), allowing the gut to straighten after the cranial surgery.

**Consequences of the surgical separation of the mesentery from the gut.**

- The gut and cranial mesentery repair their attachment, leading to some regions of normal looping.

**Quantitative analysis of gut looping.**

- The loop period increases as the gut tube grows faster than the mesentery sheet.
- The loop radius decreases as the mesentery shrinks.
- The number of loops increases as the gut tube grows faster than the mesentery sheet.

**Conclusion.**

- The findings suggest that the gut loops are influenced by the mechanical properties of the mesentery and the growth rates of the gut and mesentery.

T. Savin et al., Nature 476, 57 (2011)
Synthetic analog of guts

Rubber model of guts

Chick guts at E12

What is the wavelength of this oscillations?
Compression of stiff tube on soft elastic mesentery sheet

assumed profile
\[ h(s) = h_0 \cos\left(\frac{2\pi s}{\lambda}\right) \]

amplitude of wrinkles
\[ h_0 = \frac{\lambda}{\pi} \sqrt{\frac{\Delta}{L}} = \frac{\lambda \sqrt{\epsilon}}{\pi} \]
deformation of the soft mesentery decays exponentially away from the surface
\[ h(s, y) \approx h_0 \cos\left(\frac{2\pi s}{\lambda}\right)e^{-y/\lambda} \]

bending energy of stiff tube
\[ U_b \sim L \times \kappa_t \times \frac{1}{R^2} \sim L \times E_t I_t \times \frac{h_0^2}{\lambda^4} \sim \frac{L E_t I_t \epsilon}{\lambda^2} \]
deformation energy of soft mesentery
\[ U_m \sim A \times E_m d \times \epsilon_m^2 \sim L \lambda \times E_m d \times \frac{h_0^2}{\lambda^2} \sim L E_m d \lambda \epsilon \]

minimize total energy \((U_b + U_m)\) with respect to \(\lambda\)
\[ \lambda \sim \left(\frac{E_t I_t}{E_m d}\right)^{1/3} \]
bending stiffness of tube
\[ \kappa_t = E_t I_t \]
\[ \kappa_t \propto E_t \left(r_0^4 - r_i^4\right) \]
Wavelength of oscillations in guts

animal data, rubber model, computer simulations

\[ \frac{36}{r_0} \left( \frac{E_t I_t}{E_m d} \right)^{1/3} \]

**Figure 5**

Comparison of the scaled loop radius, predicted values of our scaling theory and computational model. In all cases, the pre-

**Discussion**

Finally, to test our theoretical model with a non-avian example, we

**Table 2**

<table>
<thead>
<tr>
<th>Species and stage</th>
<th>Stage</th>
<th>Growth strain (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chick E12</td>
<td>0.1</td>
<td></td>
</tr>
<tr>
<td>Quail E12</td>
<td>0.3</td>
<td></td>
</tr>
<tr>
<td>Finch E13</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>Mouse E16.5</td>
<td>0.8</td>
<td></td>
</tr>
</tbody>
</table>

The gut tube and the dorsal mesentery were separated

**Figure 6**

Morphometry of quail, finch and mouse gut looping pat-

**Figure 7**

The simplicity of the mechanical origin in the diversity in gut loop-

**Figure 8**

Game Bird. All eggs were incubated at 37.5

**Figure 9**

The developing intestine is a simple, elongated, tubular structure that

**Figure 10**

Gut looping patterns in the chick, quail, finch and mouse (to scale) show

**Figure 11**

Physical model of gut looping

**Figure 12**

We compared the gut looping patterns of the chick with those of the

**Figure 13**

Quantitative biomechanical perspective to the mostly metaphoric

**Figure 14**

through the process of looping morphogenesis. Using a combination

**Figure 15**

is stereotypically and reproducibly folded into a compact organ

**Figure 16**

of the mesentery sheet. The bending energy of the tube per

**Figure 17**

ratio of the contour length,

**Figure 18**

in shape to the looping patterns seen

**Figure 19**

period (Fig. 3a). The geometry of the growing gut is characterized by

**Figure 20**

= \frac{L}{R} \approx \lambda

**Figure 21**

Mouse E16.5 Experimental observation 6.0

**Figure 22**

within individual organisms and across species. We thus bring a

**Figure 23**

this study, suggesting that both growth and the properties of tissues

**Figure 24**

higher mismatch strain, producing tightly coiled loops, as seen in

**Figure 25**

growth rates, tissue geometry and elasticity of the gut–mesentery

**Figure 26**

nature.

**Figure 27**

on which selection has acted to achieve the looping patterns found in

**Figure 28**

morphogenesis opens the door to future studies of the genes involved

**Figure 29**

of quantitative experiments, computations and scaling arguments, we

**Figure 30**

between the two tissues. Embryos were re-incubated until E12, when they were collected to

**Figure 31**

two tissues. Embryos were re-incubated until E12, when they were collected to

**Figure 32**

are reported for chick at E8, E12 and E16; quail at E12 and E15; finch at E10 and
Compression of soft elastic material

When soft elastic material is compressed by more than 35% surface forms sharp creases. This is effect of nonlinear elasticity!
Swelling of thin membranes on elastic substrates

Swelling of gels

- stiff membrane, soft substrate
- soft membrane, stiff substrate
- soft membrane, soft substrate

swelling

wrinkles

creases

wrinkles + creases

Materials and Methods

We use a custom finite element method to minimize the elastic energy. We focus on the growth of the gray matter relative to the white matter, which leads to the appearance of wrinkles and sulci. By varying the cross-linking density of the materials, we can prepare samples with different ratios of the moduli of the gray and white matter. When the ratio is close to unity, we observe patterns reminiscent of sulci and gyri in the brain.

Mathematical model in two dimensions. We start with a rectangular domain consisting of a layer of gray matter on top of a deep layer of white matter, both having the same uniform shear modulus. We confirm this picture and are shown in Fig. 1.

Later on we denote the magnitude of expansion. When the outer layer is stiffer, the gray matter will both make the tissues almost incompressible. To apply a tangential growth profile, we consider the deformation gradient, which is the deformation gradient, the deformation gradient, and the bulk modulus, and are given relative to the undeformed thickness of the gray matter, and the deformed thickness of the gray matter, and the unfolded thickness of the gray matter. (Eq. 1)

The formation of a minimal sulcus can be observed as the 2D sulci form wrinkled regions. The widths and depths, as a function of the modulus ratio, are given in Fig. S1 and are consistent with observations in developing ferret brains.

The scale dimensions soften the simulated sulci (solid lines) as a function of the modulus ratio. The width and depth increase further for simplicity in the non-colored areas. Grid lines correlate with observations in lightly sulcified brains such as the porcupine. If the two layers have similar moduli, the gray matter will both form cusped folds largely internal to the gray matter and resemble the folds in lightly sulcified brains such as the porcupine.

When the outer layer is softer, the gray matter will both form wrinkles, which resemble sulci and gyri in the brain. The folding can only occur in the direction of swelling. The folding can only occur in the direction of swelling. The folding can only occur in the direction of swelling. The folding can only occur in the direction of swelling.

Coloring shows radial and circumferential tensile stress in the left and right sulci, respectively, and the bulk stress. Compressive stress is indicated with observation in developing ferret brains. T. Tallinen et al., PNAS 111, 12667 (2014)
Migration of neurons to the cortex leads to swelling of gray matter!
Brains for various organisms

Brains of different sizes are illustrated in the diagram. The tangential expansion $g^2$ is plotted against the brain size $R/T$.

- **Swelling of gel membranes**
- **Computer simulations**
- **Gyrification index** $GI = \frac{\text{area of brain surface}}{\text{area of convex hull}}$

**Measurements of brain parameters**

- $R$: brain size
- $T$: thickness of gray matter

**Tangential expansion** $g = \frac{\text{contour length of gray matter}}{\text{length of circular section}}$

**Supporting Information**

www.brainmuseum.org
Power law scaling for the brain size of various organisms

\[
\text{white matter volume} \propto \left( \frac{\text{gray matter volume}}{\text{gray matter volume}} \right)^{1.23}
\]

another scaling relation

\[
\text{gray matter thickness} \propto \left( \frac{\text{gray matter volume}}{\text{gray matter volume}} \right)^{0.10}
\]

Note: power law scalings of various quantities among organisms are very common!

Brain malformations

- lissencephaly
- pachygyria
  (small number of larger gyri)

- polymicrogyria
  (large number of smaller gyri)

Reduced neuronal migration to cortex

Gray matter is thicker and it swells less!

Typically gray matter has only four rather than six layers in the affected areas.