Double Rainbow

primary rainbow (1 internal reflection)

secondary rainbow (2 internal reflections)

Man sees rainbow. red at top, blue at bottom

Primary Bow viewing angles: from sun - to droplet - to observer’s eye = 40° to 42°
Secondary Bow viewing angles: from sun - to droplet - to observer’s eye = 52° to 54°
colors are reversed in the secondary bow!
Further reading about structural colors and photonic crystals

http://ab-initio.mit.edu/book/
Why do we get wrinkled surfaces?

Fingers after being exposed to water for some time

Old apple

Brain

Rising dough
Compression of stiff thin sheets on liquid and soft elastic substrates

Liquid substrate

Elastic substrate

10 µm thin sheet of polyester on water

~10 µm thin PDMS (stiffer) sheet on PDMS (softer) substrate

\[ \lambda_0 = 1.6 \text{ cm} \]

\[ \lambda_0 = 70 \mu\text{m} \]


Buckling vs wrinkling

Compressed thin sheets buckle

Compressed thin sheets on liquid and soft elastic substrates wrinkle

In compressed thin sheets on liquid and soft elastic substrates global buckling is suppressed, because it would result in very large energy cost associated with deformation of the liquid or soft elastic substrate!
Brief intro to mechanics:

**Young’s modulus**

Hooke’s law (small deformations)

\[
\frac{F}{A} = E \frac{\Delta L_z}{L_z}
\]

**normal stress:** \( \sigma = \frac{F}{A} \)

**Young’s modulus:** \( E \)

**normal strain:** \( \epsilon = \frac{\Delta L_z}{L_z} \)

Elastic energy of deformation

\[
U = \frac{1}{2} V E \epsilon^2
\]

**element volume:** \( V = L_x L_y L_z \)
Young’s modulus of materials

http://www-materials.eng.cam.ac.uk/mpsite/physics/introduction/
Typically material shrinks (expands) in the transverse direction of the axial tension (compression)!

\[ \nu = -\frac{\varepsilon_x}{\varepsilon_z} = -\frac{\varepsilon_y}{\varepsilon_z} \]

\[ \varepsilon_z = \frac{\sigma_z}{E} \]

normal strains: \( \varepsilon_i = \frac{\Delta L_i}{L_i} \)
Effective negative Poisson’s ratio for structures

Certain structures behave like they have effective negative Poisson’s ratio, even though they are made of materials with positive Poisson’s ratio!
**Bulk modulus**

**undeformed material element**

**hydrostatic stress**

**Hooke’s law**
(small deformations)

\[
\frac{\Delta V}{V} = -\frac{p}{K}
\]

**hydrostatic stress:** \( p \)

**bulk modulus:** \( K = \frac{E}{3(1-2\nu)} \)

**volumetric strain:** \( \frac{\Delta V}{V} \approx 3 \frac{\Delta L}{L} \)

**Elastic energy of deformation**

\[
U = \frac{1}{2} VK \left( \frac{\Delta V}{V} \right)^2 \approx VE \left( \frac{\Delta L}{L} \right)^2
\]
Shear

Note: shear stress does not change the volume of material element!

undeformed material element

undeformed material element

undeformed material element

undeformed material element

undeformed material element

Hooke’s law
(small deformations)

shear stress:

\[ \tau = \frac{F}{A} \]

shear modulus:

\[ G = \frac{E}{2(1 + \nu)} \]

shear strain:

\[ \gamma = \arctan \left( \frac{\Delta}{L_z} \right) \]

\[ \gamma \approx \frac{\Delta}{L_z} \]

Elastic energy of deformation

\[ U = \frac{1}{2} V G \gamma^2 \sim V E \left( \frac{\Delta}{L_z} \right)^2 \]

element volume:

\[ V = L_x L_y L_z \]
Arbitrary deformation of 3D solid element

Arbitrary deformation can be decomposed to the volume change and the shear deformation.

\[ U = U_{\text{bulk}} + U_{\text{shear}} \]
In plane deformations of thin sheets

**undeformed square patch of thin sheet**

- **patch area** \( A = L^2 \)

**isotropic deformation**

- 2D bulk modulus
  \[ B = \frac{Et}{2(1 - \nu)} \]

- 2D shear modulus
  \[ \mu = Gt = \frac{Et}{2(1 + \nu)} \]

**shear deformation**

- 2D bulk modulus
  \[ B = \frac{Et}{2(1 - \nu)} \]

- 2D shear modulus
  \[ \mu = Gt = \frac{Et}{2(1 + \nu)} \]

**anisotropic stretching**

- \( \epsilon_1, \epsilon_2 \ll 1 \)

(shearing can be interpreted as anisotropic stretching)

**sheet thickness** \( t \)

**Young’s modulus** \( E \)

**Poisson’s ratio** \( \nu \)
Curvature of surfaces

(A) curvature of space curves

\[ \frac{1}{R} = \frac{h''}{(1 + h'^2)^{3/2}} \approx h'' \]

R. Phillips et al., Physical Biology of the Cell

(B) curvature tensor for surfaces

\[ K_{ij} \approx \begin{pmatrix} \frac{\partial^2 h}{\partial x^2} & \frac{\partial^2 h}{\partial x \partial y} \\ \frac{\partial^2 h}{\partial x \partial y} & \frac{\partial^2 h}{\partial y^2} \end{pmatrix} \]

maximal and minimal curvatures (principal curvatures) correspond to the eigenvalues of curvature tensor
Surfaces of various principal curvatures

sphere
\[ \frac{1}{R_1} = \frac{1}{R_2} = \frac{1}{r} \]

cylinder
\[ \frac{1}{R_1} = \frac{1}{r} \]
\[ \frac{1}{R_2} = 0 \]

potato chips = “saddle”
\[ \frac{1}{R_1} > 0 \]
\[ \frac{1}{R_2} < 0 \]
Bending energy cost for thin sheets

undeformed thin sheet (thickness \( t \))

deformed thin sheet

\[
U = \int dA \left[ \frac{\kappa}{2} \left( \frac{1}{R_1} + \frac{1}{R_2} \right)^2 + \kappa_G \frac{1}{R_1 R_2} \right]
\]

\[
U \approx \int dx dy \left[ \frac{\kappa}{2} \left( \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right)^2 + \kappa_G \det \left( \frac{\partial^2 h}{\partial x_i \partial x_j} \right) \right] \quad x_i, x_j \in \{x, y\}
\]

**bending rigidity (flexural rigidity)** \( \kappa = \frac{Et^3}{12(1 - \nu^2)} \)

**Gauss bending rigidity** \( \kappa_G = \frac{Et^3}{12(1 + \nu)} \)
Compressing stiff thin sheets on liquid and soft elastic substrates

![Diagram showing compression of stiff thin sheets on liquid and soft elastic substrates.](image)

- **Liquid substrate**
  - 10 µm thin sheet of polyester on water
  - \( \lambda_0 = 1.6 \text{ cm} \)

- **Elastic substrate**
  - ~10 µm thin PDMS (stiffer) sheet on PDMS (softer) substrate
  - \( \lambda_0 = 70 \mu\text{m} \)

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Compression of stiff thin membranes on liquid substrates

Consider the energy cost for two different scenarios:
1. thin membrane is compressed (no bending)
2. thin membrane is wrinkled (no compression) + additional potential energy of liquid
Compression of stiff thin membranes on liquid substrates

\[ U_c \sim A \times E_m d \times \epsilon^2 \]

膜面积

\[ A = WL \]

膜3D杨氏模量

\[ E_m \]

应变

\[ \epsilon = \frac{\Delta}{L} \]

液体密度

\[ \rho \]

Note: upon compression the liquid surface also raises, but we will measure the potential energy relative to this new height!
Compression of stiff thin membranes on liquid substrates

assumed profile

\[ h(s) = h_0 \cos(2\pi s / \lambda) \]

projected length assuming that membrane doesn’t stretch

\[ L - \Delta = \int_0^L ds \sqrt{1 - h'(s)^2} \approx \int_0^L ds \left( 1 - h'(s)^2 / 2 \right) \approx L \left( 1 - \frac{\pi^2 h_0^2}{\lambda^2} \right) \]

amplitude of wrinkles

\[ h_0 = \frac{\lambda}{\pi} \sqrt{\frac{\Delta}{L}} = \frac{\lambda \sqrt{\epsilon}}{\pi} \]

bending energy of stiff membrane

\[ U_b \sim A \times \kappa \times \frac{1}{R^2} \sim A \times E_md^3 \times \frac{h_0^2}{\lambda^4} \sim \frac{AE_md^3 \epsilon}{\lambda^2} \]

potential energy of liquid

\[ U_p \sim m \times g \times \Delta h \sim \rho \times Ah_0 \times g \times h_0 \sim A \rho g \lambda^2 \epsilon \]

minimize total energy \((U_b + U_p)\) with respect to \(\lambda\)

\[ \lambda \sim \left( \frac{E_md^3}{\rho g} \right)^{1/4} \]

\[ U_b, U_p \sim A \epsilon \sqrt{E_md^3 \rho g} \]
Compression of stiff thin membranes on liquid substrates

\[ U_c \sim A \times E_m d \times \epsilon^2 \]

\[ U_b, U_p \sim A \epsilon \sqrt{E_m d^3 \rho g} \]

- Wrinkles are stable above the critical strain \( \epsilon > \epsilon_c \sim \sqrt{\frac{\rho g d}{E_m}} \)
- Wavelength of wrinkles \( \lambda \sim \left( \frac{E_m d^3}{\rho g} \right)^{1/4} \)
- Amplitude of wrinkles at the critical strain \( h_0^* \sim \lambda \sqrt{\epsilon_c} \sim d \)
Compression of stiff thin membranes on liquid substrates

scaling analysis

\[ \lambda \sim \left( \frac{E_m d^3}{\rho g} \right)^{1/4} \]

exact result

\[ \lambda = 2\pi \left( \frac{\kappa}{\rho g} \right)^{1/4} \]

\[ \kappa = \frac{E_m d^3}{12(1 - \nu^2_m)} \]

Compression of stiff thin membranes on liquid substrates

How to go beyond the simple scaling analysis to determine the nonlinear post-buckling behavior?

Find shape profile $h(s)$ that minimizes total energy

$$U_b + U_p = W \int_0^L \frac{ds}{2} \left[ \frac{\kappa h''^2}{(1 + h'^2)^3} + \rho g h^2 \sqrt{1 - h'^2} \right]$$

subject to constraint

$$L - \Delta = \int_0^L ds \sqrt{1 - h'^2}$$
Compression of stiff thin membranes on liquid substrates

Comparison between theory (infinite membrane) and experiment

\[ h/\lambda \quad \epsilon = 0.15 \]

\[ \epsilon = 0.30 \]

\[ \epsilon = 0.80 \]

\[ s/\lambda \]


Compression of stiff thin membranes on soft elastic substrates

Consider the energy cost for two different scenarios:

1.) thin membrane is compressed (no bending)

2.) thin membrane is wrinkled (no compression)
   additional elastic energy for deformed substrate
Compression of stiff thin membranes on soft elastic substrates

Compression energy of thin membrane

\[ U_c \sim A \times E_m d \times \epsilon^2 \]

membrane area

\[ A = W L \]

membrane 3D Young’s modulus

\[ E_m \]

strain

\[ \epsilon = \frac{\Delta}{L} \]

substrate 3D Young’s modulus

\[ E_s \]

Note: soft elastic substrate is also compressed, but we will measure the substrate elastic energy relative to this base value!
assumed profile

\[ h(s) = h_0 \cos(2\pi s / \lambda) \]

deformation of the soft substrate decays exponentially away from the surface

\[ h(s, y) \approx h_0 \cos(2\pi s / \lambda) e^{-2\pi y / \lambda} \]

amplitude of wrinkles

\[ h_0 = \frac{\lambda}{\pi} \sqrt{\frac{\Delta}{L}} = \frac{\lambda \sqrt{\epsilon}}{\pi} \]
Compression of stiff thin membranes on soft elastic substrates

assumed profile

\[ h(s) = h_0 \cos\left(\frac{2\pi s}{\lambda}\right) \]

amplitude of wrinkles

\[ h_0 = \frac{\lambda}{\pi} \sqrt{\frac{\Delta}{L}} = \frac{\lambda \sqrt{\varepsilon}}{\pi} \]

deformation of the soft substrate decays exponentially away from the surface

\[ h(s, y) \approx h_0 \cos\left(\frac{2\pi s}{\lambda}\right) e^{-2\pi y/\lambda} \]

bending energy of stiff membrane

\[ U_b \sim A \times \kappa \times \frac{1}{R^2} \sim A \times E_m d^3 \times \frac{h_0^2}{\lambda^4} \sim \frac{A E_m d^3 \varepsilon}{\lambda^2} \]

deformation energy of soft substrate

\[ U_s \sim V \times E_s \times \epsilon_s^2 \sim A\lambda \times E_s \times \frac{h_0^2}{\lambda^2} \sim A E_s \lambda \varepsilon \]

minimize total energy \((U_b + U_s)\) with respect to \(\lambda\)

\[ \lambda \sim d \left(\frac{E_m}{E_s}\right)^{1/3} \]

\[ U_b, U_s \sim A d \varepsilon \left(E_s^2 E_m\right)^{1/3} \]
Compression of stiff thin membranes on soft elastic substrates

\[ U_c \sim A \times E_m d \times \epsilon^2 \]

\[ U_b, U_s \sim Ad\epsilon \left( E_s^2 E_m \right)^{1/3} \]

wavelength of wrinkles

\[ \epsilon > \epsilon_c \sim \left( \frac{E_s}{E_m} \right)^{2/3} \]

amplitude of wrinkles at the critical strain

\[ \lambda \sim d \left( \frac{E_m}{E_s} \right)^{1/3} \]

wrinkles are stable for large strains

\[ h_0^* \sim \lambda \sqrt{\epsilon_c} \sim d \]
Compression of stiff thin membranes on liquid and soft elastic substrates

The wavelength of wrinkles on liquid substrates is given by:

\[ \lambda = 2\pi \left( \frac{\kappa}{\rho g} \right)^{1/4} \]

And the wavelength of wrinkles on soft elastic substrates is:

\[ \lambda = 2\pi \left( \frac{3\kappa}{E_s} \right)^{1/3} \]

Compression of stiff thin membranes on soft elastic substrates

In order to explain period doubling (quadrupling, …) one has to take into account the full nonlinear deformation of the soft substrate!