Shapes of swelling sheets

Self-folding origami
Mechanics of growing sheets

Growth defines preferred metric tensor $g_{ij}$, and preferred curvature tensor $K_{ij}$.

$$g'_{ij} = \frac{\partial \vec{r}'}{\partial x^i} \cdot \frac{\partial \vec{r}'}{\partial x^j}$$

$$K'_{ij} = \sum_k (g'^{-1})_{ik} \left( \vec{r}' \cdot \frac{\partial^2 \vec{r}'}{\partial x^k \partial x^j} \right)$$

strain tensors

$$u_{ij} = \frac{1}{2} \sum_k (g^{-1})_{ik} (g'_{kj} - g_{kj})$$

$$b_{ij} = K'_{ij} - K_{ij}$$

The equilibrium membrane shape $\vec{r}'(x^1, x^2)$ corresponds to the minimum of elastic energy:

$$U = \int \left( \sqrt{g} dx^1 dx^2 \right) \left[ \frac{1}{2} \lambda \left( \sum_i u_{ii} \right)^2 + \mu \sum_{i,j} u_{ij} u_{ji} + \frac{1}{2} \kappa (\text{tr}(b_{ij}))^2 + \kappa_G \det(b_{ij}) \right]$$

Growth can independently tune the metric tensor $g_{ij}$ and the curvature tensor $K_{ij}$, which may not be compatible with any surface shape that would produce zero energy cost!

Zero energy shape exists only when preferred metric tensor $g_{ij}$ and preferred curvature tensor $K_{ij}$ satisfy Gauss-Codazzi-Mainardi relations!
Mechanics of growing membranes

One of the Gauss-Codazzi-Mainardi equations (Gauss's Theorema Egregium) relates the Gauss curvature to metric tensor $\det(K'_{ij}) = \mathcal{F}(g'_{ij})$

The equilibrium membrane shape $\vec{r}'(x^1, x^2)$ corresponds to the minimum of elastic energy:

$$U = \int (\sqrt{g} dx^1 dx^2) \left[ \frac{1}{2} \lambda \left( \sum_i u_{ii} \right)^2 + \mu \sum_{i,j} u_{ij} u_{ji} + \frac{1}{2} \kappa (\text{tr}(b_{ij}))^2 + \kappa_G \det(b_{ij}) \right]$$

scaling with membrane thickness $d$

$\lambda, \mu \sim Ed$

$\kappa, \kappa_G \sim Ed^3$

For very thin membranes the equilibrium shape matches the preferred metric tensor to avoid stretching, compressing and shearing. This also specifies the Gauss curvature!

$g'_{ij} = g_{ij}$

$\det(K'_{ij}) = \mathcal{F}(g_{ij})$
Shaping of gel membranes by differential shrinking

Computer software controls valves to inject a predefined time depend concentration of NIPA polymers in water solution.

At higher temperatures gel becomes hydrophobic and expels some water. Shrinking depends on the concentration of NIPA polymers.

Frozen NIPA concentration profile

\[ C(r) \]

\[ T = 22^\circ C \]

Active cross-linkers (APS) polymerize the polymer solution within one minute, before polymers get a chance to diffuse around.

\[ T = 45^\circ C \]

"Activation" of the metric in hot water

Cross-linking of polymers result in a solid gel

polymer solution  \[\rightarrow\]  solid gel

adding active cross-linkers

cross-linkers

Note: some cross-linkers can be chemically activated by UV light exposure. Duration of UV light exposure controls the degree of cross-linking and therefore the Young’s modulus $E$ for gels.
Shaping of gel membranes by differential shrinking

Shrinking of gels at $T=45^\circ C$

Note that shrinking is uniform throughout the gel thickness and there is no preferred curvature

$$K_{ij} = 0$$

Shaping of gel membranes by differential shrinking

![Graph showing concentration C in % vs. radius r (mm)]

- **Shrinking of sheets**
  - a
  - b
  - c
  - d

- **Shrinking of tubes**
  - e
  - f
  - g
  - h

**5. Some results and interpretations**

- The concentration of the gel in the peripheries of the discs influences the shrinking properties. For discs with positive Gaussian curvature, the shrinking is slower with increasing concentration, while for discs with negative Gaussian curvature, the shrinking is faster than linear with decreasing concentration.

**5.1 Rectangle geometry**

- The shrinking properties of the tubes are compared with those of the discs to illustrate the effect of different curvatures on the shrinking process.

**References**

Shaping of gel membrane properties by lithography

Thin film of polymer solution with premixed inactive cross-linkers

UV light activates cross-linkers. Time of UV light exposure determines the degree of polymer cross-linking.

Thickness 7-17 μm

Swelling in liquid: more cross-linked regions swell less; less cross-linked regions swell more

Non cross-linked polymer solution is washed away

Ω<sub>low</sub> and Ω<sub>high</sub>

J. Kim et al., Science 335, 1201 (2012)
Halftoning

local area fraction of the low swelling regions

\[
\phi_{\text{low}} = \frac{\Delta A_{\text{low}}}{\Delta A_{\text{low}} + \Delta A_{\text{high}}} = \frac{\pi}{2\sqrt{3}} \left( \frac{d}{a} \right)^2
\]

Effective swelling \( \Omega \) can be estimated from local force balance as

\[
\frac{\phi_{\text{low}} + \alpha(1 - \phi_{\text{low}})}{\Omega^{1/2}} = \frac{\phi_{\text{low}}}{\Omega_{\text{low}}^{1/2}} + \frac{\alpha(1 - \phi_{\text{low}})}{\Omega_{\text{high}}^{1/2}}
\]

\( T = 22^\circ \text{C} \)

swelling depends on \( T \)

J. Kim et al., Science 335, 1201 (2012)
Shaping of gel membrane properties by halftone lithography

For thin membranes the target Gauss curvature is

\[
\det(K'_{ij}(x, y)) = -\frac{\nabla^2(\ln \Omega(x, y))}{2\Omega(x, y)}
\]

Inverse problem can be solved with conformal maps.

J. Kim et al., Science 335, 1201 (2012)
Shaping of gel membrane properties by halftone lithography

saddle (Sa)  cone with excess angle (Ce)  spherical cap (Sp)  cone with deficit angle (Cd)

Enneper’s minimal surfaces (H=0)

swelling profiles

H - mean curvature  K - Gauss curvature

J. Kim et al., Science 335, 1201 (2012)
Temperature controls swelling and thus the deformed shape

Note different intermediate shapes! By slowly varying the temperature we stay in a local energy minimum!

J. Kim et al., Science 335, 1201 (2012)
Gaussian curvature does not uniquely specify the shape!

target shape

swelling pattern

conformal map

this bump buckled on the wrong side

target Gauss curvature

swelling
3D printing anisotropic hydrogels

3D printed solution includes polymers, inactive cross-linkers and nanofibrillated cellulose during printing shear stresses in fluid align nanofibrillated cellulose in the direction of flow.

This procedure produces anisotropic elastic material with Young’s moduli:

- Direction of fibers: $E_\parallel \approx 40 \text{ kPa}$
- Orthogonal direction: $E_\perp \approx 20 \text{ kPa}$

After printing the cross-linkers are activated with UV light.

A. S. Gladman et al., *Nat. Materials* 15, 413 (2016)
Anisotropic swelling of hydrogels

After the hydrogel is immersed in water it swells due to absorption of water. Swelling is larger in direction orthogonal to nanofibrillated cellulose.

The inspiration for this came from plants, where the anisotropy in swelling upon changes in humidity is due to directed fibers.
Shaping hydrogels via anisotropic swelling

3D printed patterns of hydrogels

- transformed shapes after swelling
  - positive Gauss curvature
  - negative Gauss curvature
  - bending of long strip
  - twisting of long strip

A. S. Gladman et al., Nat. Materials 15, 413 (2016)
Shaping hydrogels via anisotropic swelling

“curling of leaves”

“twisting of leaves”
Shaping hydrogels via anisotropic swelling

The degree of swelling can be controlled via temperature!

Shaping hydrogels via anisotropic swelling

target shape: calla lily flower

3D printed hydrogel

3D printer in action

swollen hydrogel

A. S. Gladman et al., Nat. Materials 15, 413 (2016)
Shaping hydrogels via anisotropic swelling

target shape: orchid *Dendrobium helix*

3D printer in action

3D printed hydrogel

swollen hydrogel

A. S. Gladman et al., Nat. Materials 15, 413 (2016)
Mimosa pudica = “Touch-me-not plant”

In response to touch plant releases certain chemicals and changes the osmotic environment for cells near the base of touched leaves. As a consequence these cells lose water and their shrinking causes the folding of leaves.
Origami

Japanese for ori=fold, gami=paper
Can we make a self-folding origami?

https://www.youtube.com/watch?v=GANW-KU2yn4
Making a fold with swelling of gels

**Mountain fold**

- Swelling of yellow gel

**Valley fold**

- Swelling of yellow gel
Self folding origami with gel swelling

Pattern of valley folds

Intermediate layer

Pattern of mountain folds

Width of the “cuts” determines the folding angle

Randlett’s flapping bird

Mountain

Valley

Temperature controls swelling and thus the folding of origami

Top view of self-folding origami

a

22 °C  55 °C

b

22 °C  55 °C

Biodegradable microgrippers for robotic surgeries

Temperature regulates opening/closing of microgrippers

Position of microgrippers can be controlled with magnets

Biopsy of biological tissues

T.G. Leong et al., PNAS 106, 703 (2009)
Origami for satellite solar panels

https://www.youtube.com/watch?v=3E12uju1vgQ
Origami for shielding telescopes for detection of exoplanets

Shield is used to block the strong light coming from a star, which enables the telescope to detect faint signals from planets orbiting the star.

https://www.ted.com/speakers/jeremy_kasdin
Shrinky-Dinks

Shrinky-Dinks are sheets made of optically transparent, pre-strained polystyrene that shrink if heated to the glass transition temperature.

Localized heating and shrinking of Shrinky-Dinks can be achieved with patterning of black ink that absorb light.

Folding angle can be controlled with the width of ink and with the exposure time of light.

Shrinky-Dinks origami

size ~ cm

Sequential folding of Shrinky-Dinks origami

Different ink colors have different absorption spectra for red, green and blue light.

Sequential folding of Shrinky-Dinks origami

The order of folding corresponds to the amount of absorbed blue light \((\text{black} > \text{red} > \text{walnut})\)

Note: red ink is thicker than the walnut ink!

Sequential folding of Shrinky-Dinks origami

Converting 2D sheets to 3D objects sequentially using only light
Self-folding robots (in 4 min)

S. Felton et al., Science 345, 644 (2014)
Robot assembly

Chemical etching of copper outside the ink mask

Laser cutting of layers

Gluing of layers

“Shrinky-Dinks”

Crank Arm Pin Motors Alignment Tab Locking Tab

installment of electrical components, motors, and batteries

S. Felton et al., Science 345, 644 (2014)
Folding of robot

electric current through patterned copper network locally heats up and shrinks the “Shrinky-Dinks” layer

S. Felton et al., Science 345, 644 (2014)
Structures with mechanisms

Structures composed of bars and hinges, which have fewer constraints than degrees of freedom, have specific mechanisms (=modes of deformations)

scissor lift

precise robotic surgeries

changing direction of motion

amplifying/reducing amplitude of motion
Crank slider mechanism

Crank slider mechanism converts linear to rotary motion!

Crank slider mechanism in car engines

https://en.wikipedia.org/wiki/Crank_(mechanism)
Robot actuation

Sequential folding enables locking of the crank arm to the robot structure

Rotary motor moves the crank arm, which controls the movement of robot legs via a specific structure mechanism