Wrinkled surfaces
Compression of stiff thin membranes on liquid substrates

\[ U_c \sim A \times E_m d \times \epsilon^2 \]

\[ U_b, U_p \sim A \epsilon \sqrt{E_m d^3 \rho g} \]

wrinkles are stable above the critical strain \( \epsilon > \epsilon_c \sim \sqrt{\frac{\rho g d}{E_m}} \)

wavelength of wrinkles \( \lambda \sim \left( \frac{E_m d^3}{\rho g} \right)^{1/4} \)

amplitude of wrinkles at the critical strain \( h_0^* \sim \lambda \sqrt{\epsilon_c} \sim d \)
Compression of stiff thin membranes on soft elastic substrates

Consider the energy cost for two different scenarios:

1.) thin membrane is compressed (no bending)

2.) thin membrane is wrinkled (no compression)
additional elastic energy for deformed substrate
Compression of stiff thin membranes on soft elastic substrates

Compression energy of thin membrane

\[ U_c \sim A \times E_m d \times \epsilon^2 \]

- **membrane area**
  \[ A = WL \]
- **membrane 3D Young’s modulus**
  \[ E_m \]
- **strain**
  \[ \epsilon = \frac{\Delta}{L} \]
- **substrate 3D Young’s modulus**
  \[ E_s \]

\[ E_s \ll E_m \]

Note: soft elastic substrate is also compressed, but we will measure the substrate elastic energy relative to this base value!
Compression of stiff thin membranes on soft elastic substrates

assumed profile

\[ h(s) = h_0 \cos(2\pi s / \lambda) \]

amplitude of wrinkles

\[ h_0 = \frac{\lambda}{\pi} \sqrt{\frac{\Delta}{L}} = \frac{\lambda \sqrt{\epsilon}}{\pi} \]

deformation of the soft substrate decays exponentially away from the surface

\[ h(s, y) \approx h_0 \cos(2\pi s / \lambda) e^{-2\pi y / \lambda} \]

Compression of stiff thin membranes on soft elastic substrates

assumed profile

\[ h(s) = h_0 \cos\left(\frac{2\pi s}{\lambda}\right) \]

amplitude of wrinkles

\[ h_0 = \frac{\lambda}{\pi} \sqrt{\frac{\Delta}{L}} = \frac{\lambda \sqrt{\epsilon}}{\pi} \]

deformation of the soft substrate decays exponentially away from the surface

\[ h(s, y) \approx h_0 \cos\left(\frac{2\pi s}{\lambda}\right)e^{-2\pi y/\lambda} \]

bending energy of stiff membrane

\[ U_b \sim A \times \kappa \times \frac{1}{R^2} \sim A \times E_m d^3 \times \frac{h_0^2}{\lambda^4} \sim \frac{A E_m d^3 \epsilon}{\lambda^2} \]

deformation energy of soft substrate

\[ U_s \sim V \times E_s \times \epsilon_s^2 \sim A \lambda \times E_s \times \frac{h_0^2}{\lambda^2} \sim A E_s \lambda \epsilon \]

minimize total energy \((U_b+U_s)\) with respect to \(\lambda\)

\[ \lambda \sim d \left(\frac{E_m}{E_s}\right)^{1/3} \]

\[ U_b, U_s \sim A d \epsilon \left(\frac{E_s^2 E_m}{E_m}\right)^{1/3} \]
Compression of stiff thin membranes on soft elastic substrates

\[ U_c \sim A \times E_m d \times \epsilon^2 \]

\[ U_b, U_s \sim A d \epsilon \left( \frac{E_s^2 E_m}{E_m} \right)^{1/3} \]

- wrinkles are stable for large strains
- wavelength of wrinkles
- amplitude of wrinkles at the critical strain

\[ \epsilon > \epsilon_c \sim \left( \frac{E_s}{E_m} \right)^{2/3} \]

\[ \lambda \sim d \left( \frac{E_m}{E_s} \right)^{1/3} \]

\[ h_0^* \sim \lambda \sqrt{\epsilon_c} \sim d \]
Compression of stiff thin membranes on liquid and soft elastic substrates

\[
\lambda = 2\pi \left( \frac{\kappa}{\rho g} \right)^{1/4}
\]

wavelength of wrinkles on liquid substrates

\[
\lambda = 2\pi \left( \frac{3\kappa}{E_s} \right)^{1/3}
\]

wavelength of wrinkles on soft elastic substrates

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Compression of stiff thin sheets on liquid and soft elastic substrates

- **Liquid substrate**
  - Wrinkles are stable above the critical strain: \( \epsilon > \epsilon_c \sim \sqrt{\frac{\rho gd}{E_m}} \)
  - Wavelength of wrinkles: \( \lambda \sim \left( \frac{E_m d^3}{\rho g} \right)^{1/4} \)
  - Amplitude of wrinkles: \( h_0 \sim \lambda \sqrt{\epsilon} \)

- **Soft elastic substrate**
  - Wrinkles are stable for large strains: \( \epsilon > \epsilon_c \sim \left( \frac{E_s}{E_m} \right)^{2/3} \)
  - Wavelength of wrinkles: \( \lambda \sim d \left( \frac{E_m}{E_s} \right)^{1/3} \)
  - Amplitude of wrinkles: \( h_0 \sim \lambda \sqrt{\epsilon} \)

\( E_s \ll E_m \)
Compression of stiff thin membranes on soft elastic substrates

In order to explain period doubling (quadrupling, …) one has to take into account the full nonlinear deformation of the soft substrate!
Uniform compression of stiff thin membranes on soft elastic substrates

\[ \varepsilon_c \sim \left( \frac{E_s}{E_m} \right)^{2/3} \]

\[ \lambda \sim d \left( \frac{E_m}{E_s} \right)^{1/3} \]

The preference of our experimental system to assume the hexagonal mode at low overstress is overwhelming, despite why the hexagonal mode has always been observed to buckle with the hexagonal regions directed into the substrate, while the square mode has been shown here and previously to have lower energy in this range when the film/substrate buckles into the substrate (e.g., Lin and Yang, 2007).

It will be seen that initial spherical curvature of the film is likely to explain the two experimental observations cited above that are otherwise inexplicable when the films are taken to be flat.

An exceptionally well-organized herringbone pattern is seen in the coalescing of a neighboring pair. These triggered pairs are usually situated along a lattice line that is not parallel to that of the original pair, in order to accommodate the local stress in an equi-biaxial manner. In some cases, these coalesced

coagulated grooves link to form even longer grooves, but in general they tend to remain the product of just 2–3 hexagons. The overall result is a pattern that locally resembles a ''segmented labyrinth'', or a herringbone pattern with no global orientation, analogous to the labyrinth pattern reported for homogeneously initiated wrinkling at high overstress (Huang et al., 2005; Z1.70/C120Z1.70).

A sequence of pictures depicting the transition of the hexagonal mode to a ''segmented labyrinth'' (disorganized herringbone) pattern with different periodic wavelengths. The mechanism by which the hexagonal mode transitions to a more energy-minimizing pattern is seen in the hexagonal lattice of the original pattern, perhaps due to kinetic considerations of forming an entirely new pattern with checkerboard modes, the square mode has the lowest energy. (iii) The hexagonal mode and triangular mode have identical energy in the buckled state, to the order obtained here, and a continuous transition exists from one to the other at constant energy. (iv) Within the range of overstress considered in this paper, nonlinearity of the substrate has essentially no effect on energy. (v) The only modes whose nodal lines coincide with a specific site are the square, checkerboard, and hexagonal modes. The hexagonal mode, assuming flat films. Moreover, a new triangular mode will be identified that has precisely the same energy. (iv) Within the range of overstress considered in this paper, nonlinearity of the substrate has essentially no effect on energy. (v) The only modes whose nodal lines coincide with a specific site are the square, checkerboard, and hexagonal modes. The hexagonal mode, assuming flat films. Moreover, a new triangular mode will be identified that has precisely the same energy.

The experimental observations noted here have motivated us to look for a theoretical explanation of why the square mode is never observed for our experimental system even though under equi-biaxial stressing it has lower energy than the hexagonal mode.

Any selection observed experimentally and in determining the sign of the hexagonal deflection.


1.) compression

2.) stretching and gluing

3.) differential swelling of gels

4.) differential growth in biology

5.) differential expansion due to temperature, electric field, etc.
Compression of stiff thin membranes on a spherical soft substrates

Spherical shells are compressed by reducing internal pressure

Phase diagram of dimples/wrinkles

Hexagonal phase

Bistable phase

Labyrinth phase

characteristic wavelength is almost independent of radius R

\[ \lambda \sim d \left( \frac{E_m}{E_s} \right)^{1/3} \]

Compression of stiff thin membranes on a spherical soft substrates

Swelling of gels (red gel swells more than the green one)

Phase diagram of dimples/wrinkles

\[
\frac{\epsilon}{\epsilon_c} - 1
\]

\[
\text{Excess stress}
\]

\[
\text{Effective radius}
\]

\[
R/d
\]

\[
D. \text{ Breid and A.J. Crosby, Soft Matter 9, 3624 (2013)}
\]

Modifying radius \(R\) (fixed thickness \(d\))

\[
R = 152 \mu m \quad R = 381 \mu m \quad R = 428 \mu m
\]

Modifying membrane thickness \(d\)

\[
R = 381 \mu m
\]

Modifying swelling strain \(\epsilon\)

\[
R = 522 \mu m
\]
**Tuning drag coefficient via wrinkling**

**Drag Force**

\[ F_d = \frac{1}{2} C_D \rho u^2 A \]

- \( \rho \): air density
- \( u \): air flow speed
- \( R \): sphere radius
- \( A = \pi R^2 \): sphere cross-section area
- \( \mu \): air viscosity

\[ Re = \frac{\rho u (2R)}{\mu} \gg 1 \quad \text{Reynolds Number} \]

Depth of wrinkling is controlled via the reduction of internal pressure \( \Delta P \).

**Wrinkling reduces drag coefficient**

- **Smooth**
- **Wrinkled**

**Differential Pressure** \( \Delta P \) [Pa]

- \( \Delta P = 4800 \text{ Pa} \)
- \( \Delta P = 17000 \text{ Pa} \)
- \( \Delta P = 38000 \text{ Pa} \)
- \( \Delta P = 76000 \text{ Pa} \)

Self-cleaning property of lotus leaves

Lotus leaves repel water (hydrophobicity) due to the rough periodic microstructure

Tuning wetting angle via wrinkling

Water droplet on a flat surface

Water droplet on a wrinkled surface (wrinkling increases contact angle)


Published on 04 July 2007. Downloaded by Princeton University on 21/02/2017 01:37:01.
Tuning adhesion via wrinkling

Flat compliant surface has enhanced adhesion (larger contact area)

Wrinkling reduces adhesion (smaller contact area)

Wrinkled structures can be used for flexible electronics

B. Xu et al., Adv. Mater. 28, 4462 (2016)
How are villi formed in guts?

Villi increase internal surface area of intestine for faster absorption of digested nutrients.
Lumen patterns in chick embryo

- DAPI marks cell nuclei
- αSMA marks smooth muscle actin
- E...: age of chick embryo in days

E6: Whole gut
E8: Transverse sections of guts labeled as in (D); cultured in the presence of 10 μM FK506, drugs known to block the differentiation of smooth muscle layers. (Right) Whole-mount fluorescent images of chick midgut guts cultured under various conditions. (A) In ovo E8 guts (Fig. 2D). When E6 guts were cultured in the presence of 10 μM FK506, drugs known to block the differentiation of smooth muscle layers. (B) Transverse sections of guts labeled as in (D); cultured in the presence of 10 μM FK506, drugs known to block the differentiation of smooth muscle layers.

Stiff muscles grow slower than softer mesenchyme and endoderm layers

Radial compression due to differential growth produces striped wrinkles

Endoderm mesenchyme muscle

No muscle + circ. muscle

up until E6 Smooth
E8-E12 Ridges

A. Shyer et al., Science 342, 212 (2013)
Lumen patterns in chick embryo

Formation of longitudinal muscles at E13 produces longitudinal compression

endoderm
mesenchyme
muscle

A. Shyer et al., Science 342, 212 (2013)
Lumen patterns in chick embryo

Villi start forming at E16 because of the faster growth in valleys

The same mechanism for villi formation also works in other organisms!

E13

E15

E16

E17

A. Shyer et al., Science 342, 212 (2013)