Elastic multipole method for describing linear deformation of infinite 2D solid structures with circular holes and inclusions

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Elastic materials with holes and inclusions are important in a large variety of contexts ranging from construction material to biological membranes. More recently they have also been exploited in mechanical metamaterials, where the geometry of highly deformable structures is responsible for their unusual properties, such as the negative Poisson’s ratio, mechanical cloaking and tunable phononic band gaps. Understanding how such structures deform in response to applied external loads is thus crucial for designing novel mechanical metamaterials. Here we present a method for predicting the linear response deformation of infinite 2D solid structures with circular holes and inclusions by employing analogies with electrostatics. Just like an external electric field induces polarization (dipoles, quadrupoles, etc.) of conductive and dielectric objects, an external stress induces elastic multipoles inside holes and inclusions. Stresses generated by these induced elastic multipoles then lead to interactions between holes and inclusions, which induce additional polarization and thus additional deformation of holes and inclusions. We present a method that expands the induced polarization in a series of elastic multipoles, which systematically takes into account the interactions of inclusions and holes with external field and also between them. The results of our method show good agreement with both finite element simulations and experiments.

I. INTRODUCTION

Elastic materials with holes and inclusions have been studied extensively in materials science. Typically, the goal is to homogenize the microscale distribution of holes and inclusions to obtain effective material properties on the macroscale [1–4], where the detailed micropattern of deformations and stresses is ignored. On the other hand, it was recently recognized that the microscale interactions between proteins embedded in biological membranes can promote the assembly of ordered protein structures [5–7] and can also facilitate the entry of viral particles into cells [8]. Furthermore, in mechanical metamaterials [9] the geometry, topology and contrasting elastic properties of different materials, are exploited to obtain extraordinary functionalities, such as shape morphing [10, 11], mechanical cloaking [12–14], negative Poisson’s ratio [15–19], negative thermal expansion [20, 21], effective negative swelling [22–24], and tunable band gaps [25–27]. At the heart of these functionalities are deformation patterns of such materials with holes and inclusions. Therefore, understanding how these structures deform under applied external load is crucial for designing novel metamaterials.

Linear deformations of infinite thin plates with circular holes under external loading have been studied extensively over the years [28–32]. The solution for one hole can be easily obtained using standard techniques [33] and the solution for two holes can be constructed with the help of conformal maps and complex analysis [29]. Green demonstrated how one can construct a solution for infinite thin plates with an arbitrary number of holes [28] by expanding the Airy stress function around each hole in terms of the Michell solution for biharmonic functions [34]. However, it remained unclear how to generalize this procedure to finite structures with boundaries.

Deformations of thin membranes with infinitely rigid inclusions have also received a lot of attention, especially in the context of rigid proteins embedded in biological membranes [6–8, 35–42]. Several different approaches were developed to study the elastic and entropic interactions between inclusions, such as the multipole expansion [35], the effective field theory approach [40, 41], and homogenization [8]. Even though these articles considered membrane bending, the governing equation for the out-of-plane displacement is also biharmonic to the lowest order. Hence these methods could be adapted to investigate the in-plane deformations of plates with rigid inclusions.

In the two companion papers we generalized Green’s method [28] by employing analogies with electrostatics to describe the stress distribution and displacements of cylindrical holes and inclusions embedded either in thin elastic
FIG. 1. Induction in electrostatics and elasticity. (a,b) External electric field $E_0$ induces (a) a polarization $p$ at the center of a single conducting sphere (yellow) and (b) nonuniform charge distributions on the surface of multiple conducting spheres (yellow). The resultant electric field lines are shown in grey color. (c,d) External uniaxial compressive stress $\sigma_0$ induces (c) quadrupoles $Q$ and $P$ at the center of a circular hole embedded in an elastic matrix and (d) nonuniform charge distributions at the circumferences of multiple holes embedded in an elastic matrix. Heat maps show the von Mises stress field, where red and blue colors indicate regions of high and low stress, respectively.

plates (plane stress) or in infinitely thick elastic matrix (plane strain) that are under small external loads, which can be treated as a 2D problem with circular holes and inclusions. Just like a conductive object gets polarized in external electric field (see Fig. 1a), a deformed hole due to external load can be described with induced elastic quadrupoles (see Fig. 1c). Circular inclusions in elastic matrix under external load are analogous to dielectric objects in an external electric field. When multiple conductive objects are placed in an external electric field, the induced polarizations generate additional electric field, which leads to further charge redistribution on the surface of conductive objects (see Fig. 1b). Similarly, induced quadrupoles in deformed holes generate additional stresses in the elastic matrix, which lead to further deformations of holes (see Fig. 1d).

In this paper, we present a method to describe linear deformations of circular holes and inclusions inside an infinite 2D elastic matrix under small external loads, by systematically expanding induced polarization of each hole/inclusion in terms of elastic multipoles, which are related to terms in the Michell solution for biharmonic functions [34]. The accuracy of this method is compared against finite element simulations and experiments, and we show that the error decreases exponentially as the maximum degree of elastic multipoles is increased. In the companion paper [43], we describe how this method can be generalized to finite size structures by employing ideas of image charges that become important for holes and inclusions near boundaries.

The remaining part of the paper is organized as follows. In Section II, we review the analogy between electrostatics and 2D linear elasticity and we introduce important concepts that are borrowed from electrostatics. In Section III, we describe the method for evaluating deformation of structures with holes and inclusions under external load, which is compared against the finite element simulations and experiments. We conclude in Section IV by discussing potential extensions of this method to structures with holes and inclusions of more complex shapes.

II. ANALOGY BETWEEN ELECTROSTATICS AND 2D LINEAR ELASTICITY

The analogy between electrostatics and 2D linear elasticity can be recognized, when the governing equations are formulated in terms of the electric potential $U$ (see, e.g. [44]) and the Airy stress function $\chi$ [33], respectively, which are summarized in Table 1. The measurable fields, namely the electric field $E$ and the stress tensor field $\sigma_{ij}$, are obtained by taking spatial derivatives of these scalar functions as shown in Eqs. (1) and (2), where $\epsilon_{ij}$ is the permutation symbol ($\epsilon_{12} = -\epsilon_{21} = 1, \epsilon_{11} = \epsilon_{22} = 0$) and the summation over repeated indices is implied. The most compelling aspect of the formulations in terms of scalar functions $U$ and $\chi$, is that the Faraday’s law in electrostatics in Eq. (3) and the force balance equation in elasticity in Eq. (4) are automatically satisfied. Moreover, the governing equations
TABLE I. Comparison between equations in electrostatics and 2D linear elasticity

<table>
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<tr>
<th>Electrostatics</th>
<th>Elasticity</th>
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<td><strong>Scalar potentials</strong></td>
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<td><strong>Fields</strong></td>
<td>$E = -\nabla U$ (1)</td>
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<td><strong>Properties of scalar functions</strong></td>
<td>$\nabla \times E = -\nabla \times \nabla U = 0$ (3)</td>
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for these scalar functions take simple forms as shown in Eqs. (5) and (6). Eq. (5) describes the well known Gauss’s law, where $\rho_e$ is the electric charge density and $\epsilon_e$ is the permittivity of material. The analogous Eq. (6) in elasticity can be derived by combining Eq. (2) with constitutive relations and incompatibility conditions \[43\] \[44\], where $E$ is the 2D Young’s modulus and $\rho$ is the elastic charge density that is related to defects in the material. In the absence of electric charges ($\rho_e = 0$) the electric potential $U$ is a harmonic function (see Eq. (5)), while in the absence of defects ($\rho = 0$) the Airy stress function $\chi$ is a biharmonic function (see Eq. (6)).

When a conductive object is placed in an external electric field it gets polarized due to the redistribution of charges (see Fig. 1a). This induced polarization generates an additional electric field outside the conductive object, which can be expanded in terms of fictitious multipoles (dipole, quadrupole, etc.) located at the center of the conductive object \[44\]. Similarly, a hole or inclusion embedded in an elastic matrix gets polarized, when external load is applied (see Fig. 1a). The additional stresses in elastic matrix due to this induced polarization can again be expanded in terms of fictitious elastic multipoles (quadrupole, etc.) located at the center of the hole/inclusion. In order to demonstrate this, we first briefly present the multipoles and induction in electrostatics and we describe the meaning of their counterparts in elasticity.

### A. Monopoles (disclinations)

In electrostatics, a monopole is defined as the electric charge density distribution proportional to the Dirac delta function, i.e. $\rho_e = q\delta(\mathbf{x} - \mathbf{x}_0)$, where $q$ is the monopole charge and $\mathbf{x}_0$ denotes its position. The electric potential $U_m$ in 2D is then obtained by solving the governing equation as \[43\]

$$\Delta U_m = -q/\epsilon_e \delta(\mathbf{x} - \mathbf{x}_0) \implies U_m(\mathbf{x} - \mathbf{x}_0, q) = -q/(2\pi\epsilon_e) \ln |\mathbf{x} - \mathbf{x}_0|. \quad (7)$$

For the positive monopole charge the electric field $\mathbf{E}_m = -\nabla U_m$ is pointing radially outward (see Fig. 2a).

Similar to electrostatics we can define a monopole with charge $s$ in 2D elasticity as the charge density proportional to the Dirac delta function, i.e. $\rho = s\delta(\mathbf{x} - \mathbf{x}_0)$. Monopoles are called disclinations and their Airy stress function can be obtained by solving the governing equation as \[43\]

$$\Delta\Delta \chi_m = Es\delta(\mathbf{x} - \mathbf{x}_0) \implies \chi_m(\mathbf{x} - \mathbf{x}_0, s) = E_s/\pi^2 |\mathbf{x} - \mathbf{x}_0|^2 \ln |\mathbf{x} - \mathbf{x}_0| - 1/2. \quad (8)$$

The physical interpretation of monopoles in 2D elasticity comes from the condensed matter theory. When a wedge with angle $s$ is cut out from a 2D elastic material and the newly created boundaries of the remaining material are glued together, a positive disclination defect of charge $s$ is formed (see Fig. 2b). The negative disclination with charge $s < 0$ corresponds to the insertion of a wedge with angle $|s|$ (see Fig. 2b). The stresses generated by these operations are described with the Airy stress function in Eq. (8) \[43\] \[44\].

### B. Dipoles (dislocations)

An electrostatic dipole is formed at $\mathbf{x}_0$ when two opposite charges $\pm q$ are located at $\mathbf{x}_\pm = \mathbf{x}_0 \pm a/2$ (see Fig. 2b). The electric potential for a dipole in 2D is thus

$$U_d(\mathbf{x} - \mathbf{x}_0, p = qa) = U_m(\mathbf{x} - \mathbf{x}_+, q) + U_m(\mathbf{x} - \mathbf{x}_-, -q) \xrightarrow{|a| \to 0} \frac{p \cdot (\mathbf{x} - \mathbf{x}_0)}{2\pi\epsilon_e |\mathbf{x} - \mathbf{x}_0|^2}. \quad (9)$$


FIG. 2. Multipoles in electrostatics and 2D elasticity. (a) An electrostatic monopole of positive charge $q$ (green) generates an outward radial electric field (black lines). For a monopole of negative charge the direction of electric field is reversed. (b) In 2D elasticity a disclination defect (monopole) of charge $s$ forms upon removal ($s > 0$) or insertion ($s < 0$) of a wedge of material, where $|s|$ is the wedge angle. On the left, a wedge of angle $\pi/3$ is cut out of a triangular lattice and the resultant boundaries are glued together to form a positive disclination defect (green) with elastic charge $s = +\pi/3$, which has the coordination number 5 whereas all other points have coordination number 6. On the right, a wedge of angle $\pi/3$ is inserted into the triangular lattice to form a negative disclination defect (red) with elastic charge $s = -\pi/3$, which has the coordination number 7. (c) An electrostatic dipole $p$ is formed when a positive (green) and a negative (red) charge of equal magnitude are brought close together. The resulting electric field lines are shown with black lines. (d) In 2D elasticity a dislocation (dipole) forms upon removal or insertion of a semi-infinite strip of material of width $|b|$ and is represented with the Burger’s vector $b$. In a triangular lattice the dislocation corresponds to two adjacent disclinations of opposite charges. The two black lines indicate the positions of points before and after the removal of semi-infinite strip from the crystal. A dipole moment $d$ can be defined in the direction from the negative to positive disclination and its magnitude is the distance between the two disclinations times the magnitude of the charge of each disclination. (e) An electrostatic quadrupole $Q$ consisting of four charges at the vertices of a square with opposite charges at the adjacent vertices. The resulting electric field lines are shown with black lines. (f) In 2D elasticity a quadrupole $Q$ is analogous to the electrostatic quadrupole and is represented with four disclinations at the vertices of a square with opposite charges at the adjacent vertices. Due to the quadrupole $Q$ material expands in the direction of positive disclinations and contracts in the direction of negative disclinations, while the total area remains unchanged. In 2D elasticity a quadrupole $P$ is described with three dislocations oriented at $2\pi/3$ degrees with respect to each other. For positive (negative) quadrupole $P$ dislocations are pointing outward (inward) and they cause the surrounding material to contract (expand) isotropically.

where we introduced the dipole moment $p = qa$. \[44\]

Similarly, a dipole $d = sa$ in 2D elasticity is formed when two disclinations defects of opposite charges $\pm s$ are located at $x_{\pm} = x_0 \pm a/2$ (see Fig. 2d). Dipoles are called dislocations and their Airy stress function is \[45\]

$$\chi_d(x - x_0, d) = \chi_m(x - x_+, s) + \chi_m(x - x_-, -s) \xrightarrow{|a| \to 0} -\frac{E}{4\pi} d \cdot (x - x_0) \ln |x - x_0|. \quad (10)$$

The dislocation defect forms upon removal or insertion of a semi-infinite strip of material of width $|b|$ (see Fig. 2d). Note that dislocations are conventionally represented with the Burger’s vector $b$, which is equal to the dipole moment $d$ rotated by 90°, i.e. $b_i = \epsilon_{ij} d_j$. \[49\]
C. Quadrupoles

An electrostatic quadrupole in 2D is formed when two positive and negative charges are placed symmetrically around \(x_0\), such that charges \(q_i = q(-1)^i\) are placed at locations \(x_i = x_0 + a(\cos(\theta + i\pi/2), \sin(\theta + i\pi/2))\), where \(i \in \{0, 1, 2, 3\}\) (see Fig. 2). The electric potential of quadrupole in polar coordinates \((x-x_0 = r \cos \varphi, y-y_0 = r \sin \varphi)\) is thus

\[
U_Q(r, \varphi, Q = qa^2, \theta) = \sum_{i=0}^{3} U_m(x - x_i, q_i) \xrightarrow{a \to 0} \frac{Q \cos(2(\varphi - \theta))}{\pi \epsilon r^2},
\]

where we introduced the quadrupole moment \(Q = qa^2\).

Similarly, an elastic quadrupole \(Q\) is formed when two positive and negative disclinations are placed symmetrically around \(x_0\), such that disclinations of charges \(s_i = s(-1)^i\) are placed at locations \(x_i = x_0 + a(\cos(\theta + i\pi/2), \sin(\theta + i\pi/2))\), where \(i \in \{0, 1, 2, 3\}\) (see Fig. 2). The Airy stress function for the quadrupole \(Q\) in polar coordinates is thus

\[
\chi_Q(r, \varphi, Q = sa^2, \theta) = \sum_{i=0}^{3} \chi_m(x - x_i, s_i) \xrightarrow{a \to 0} \frac{EQ \cos(2(\varphi - \theta))}{4\pi},
\]

where we introduced the quadrupole moment \(Q = sa^2\). Due to the elastic quadrupole \(Q\), material locally expands in the \(\theta\) direction and it contracts in the orthogonal direction (see Fig. 2 and 16).

In elasticity, we can define another quadrupole \(P\) by placing three dislocation dipoles \(d_i = d(\cos(\theta + 2\pi i/3), \sin(\theta + 2\pi i/3))\) symmetrically around \(x_0\) at locations \(x_i = x_0 + a(\cos(\theta + 2\pi i/3), \sin(\theta + 2\pi i/3))\), where \(i \in \{0, 1, 2\}\) (see Fig. 2 and 16). The Airy stress function for quadrupole \(P\) is

\[
\chi_P(x - x_0, P = 3da/2) = \sum_{i=0}^{2} \chi_d(x - x_i, d_i) \xrightarrow{a \to 0} \frac{P\Delta_0 \chi_m(x - x_0)}{4\pi} = \frac{EP(1 + 2 \ln(|x - x_0|))}{4\pi},
\]

where we introduced the strength of quadrupole \(P = 3da/2\) and \(\Delta_0\) is the Laplace derivative with respect to \(x_0\). Note that the constant term in Eq. 13 does not generate any stresses and can thus be omitted. A positive (negative) quadrupole with \(P > 0\) \((P < 0)\) is related to a vacancy (interstitial) defect [47], which causes the surrounding elastic material to contract (expand) isotropically [46]. Note that a similarly defined quadrupole \(P\) in electrostatics vanishes everywhere except at the origin due to the governing Eq. 4.

D. Higher order multipoles

The procedure described in previous sections can be generalized to define higher order multipoles. In 2D the quadrupole \(Q\) is generalized by placing \(n\) positive and \(n\) negative disclinations symmetrically around \(x_0\), such that disclinations of charges \(s_i = s(-1)^i\) are placed at locations \(x_i = x_0 + a(\cos(\theta + i\pi/n), \sin(\theta + i\pi/n))\), where \(i \in \{0, 1, \ldots, 2n-1\}\). The Airy stress functions for such multipoles in polar coordinates are

\[
\sum_{i=0}^{2n-1} \chi_m(x - x_i, s_i) \xrightarrow{a \to 0} \frac{EQ(n) \cos(n(\varphi - \theta))}{4(n-1)\pi r^{n-2}},
\]

where we introduced the strength of multipoles \(Q(n) = sa^n\). The quadrupole \(P\) is generalized to higher order multipole by placing \(n\) positive and \(n\) negative quadrupoles \(P\) symmetrically around \(x_0\), such that quadrupoles of strength \(P_i = P(-1)^i\) are placed at locations \(x_i = x_0 + a(\cos(\theta + i\pi/m), \sin(\theta + i\pi/n))\), where \(i \in \{0, 1, \ldots, 2n-1\}\). The Airy stress functions for such multipoles in polar coordinates are

\[
\sum_{i=0}^{2n-1} \chi_P(x - x_i, P_i) \xrightarrow{a \to 0} \frac{EP(n) \cos(n(\varphi - \theta))}{\pi r^n},
\]

where we introduced the strength of multipole \(P(n) = Pa^n\).
E. Multipoles vs the Michell solution for biharmonic functions

The elastic multipoles introduced in previous sections are closely related to the general solution of the biharmonic equation \( \Delta \Delta \chi = 0 \) due to Michell [31], which is given in polar coordinates \((r, \varphi)\) as

\[
\chi = A_0 r^2 + B_0 r^2 \ln r + C_0 \ln r + I \varphi \\
+ (A_1 r + B_1 r^{-1} + B_1' r \varphi + C_1 r^3 + D_1 r \ln r) \cos \varphi \\
+ (E_1 r + F_1 r^{-1} + F_1' r \varphi + G_1 r^3 + H_1 r \ln r) \sin \varphi \\
+ \sum_{n=2}^{\infty} (A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2}) \cos n \varphi \\
+ \sum_{n=2}^{\infty} (E_n r^n + F_n r^{-n} + G_n r^{n+2} + H_n r^{-n+2}) \sin n \varphi.
\] 

(16)

The Michell solution above contains the Airy stress functions corresponding to multipoles located at the origin: disclination \((r^2 \ln r)\), dislocation \((r \ln r \cos \varphi, r \ln r \sin \varphi)\), quadrupole \(P\) \((\ln r)\), quadrupole \(Q\) \((\cos 2 \varphi, \sin 2 \varphi)\), as well as all higher order multipoles (see Eqs. (14, 15)). Note that the Michell solution also contains terms that increase faster than \(r^2\) far away from the origin. These terms are associated with stresses that increase away from the origin and can be interpreted as multipoles located at infinity. Due to the connection with elastic multipoles we refer to coefficients \(A_i, B_i, \ldots, H_i\) in the Michell solution as the amplitudes of multipoles.

F. Induction

As mentioned previously, external electric field induces polarization in conducting and dielectric objects. Similarly, external stress induces elastic quadrupoles inside holes and inclusions. To make this analogy precise we first demonstrate how external electric field in 2D polarizes a single conductive or dielectric disk and then we discuss how the external stress induces quadrupoles inside a circular hole or inclusion.

Let us consider a perfectly conductive disk of radius \(R\) in a uniform external electric field \(\mathbf{E} = E_0 \hat{x}\) in 2D. This electric field provides a driving force for mobile charges on a disk that are being redistributed until the resulting tangential component of the total electric field at the circumference of the disk is zero, which means that the electric potential is constant on the circumference \((r = R)\). Assuming that the electric potential is zero on the circumference of the disk and that the resultant electric field approaches the background field far away from the disk, we can solve the governing Eq. (3) with \(\rho_e = 0\) in polar coordinates to find that the electric potential is \(U^{\text{in}}(r, \varphi) = 0\) inside the conductive disk \((r < R)\) and that the electric potential \(U^{\text{out}}(r, \varphi)\) outside the conductive disk \((r > R)\) is given by [48]

\[
U^{\text{out}}(r, \varphi) = -E_0 r \cos \varphi + E_0 \frac{R^2}{r} \cos \varphi,
\] 

(17)

where the origin of coordinate system is at the center of the conductive disk. The first term in the above Eq. (17) for the electric potential \(U^{\text{out}}(r, \varphi)\) outside the conductive disk is due to the external electric field and the second term can be interpreted as the electric potential of an induced electrostatic dipole at the center of the disk (see Eq. (9) and Fig. 1a). This analysis can be generalized to a dielectric disk with dielectric constant \(\epsilon_\text{in}\) that is embedded in a material with dielectric constant \(\epsilon_\text{out}\) in a uniform external electric field. One finds that the electric potentials inside and outside the disk are then given by [48]

\[
U^{\text{in}}(r, \varphi) = -E_0 r \cos \varphi + E_0 \left(\frac{\epsilon_\text{in} - \epsilon_\text{out}}{\epsilon_\text{in} + \epsilon_\text{out}}\right) r \cos \varphi,
\] 

(18a)

\[
U^{\text{out}}(r, \varphi) = -E_0 r \cos \varphi + E_0 \left(\frac{\epsilon_\text{in} - \epsilon_\text{out}}{\epsilon_\text{in} + \epsilon_\text{out}}\right) \frac{R^2}{r} \cos \varphi.
\] 

(18b)

The first terms in both \(U^{\text{in}}\) and \(U^{\text{out}}\) correspond to the external electric field, whereas the second terms can be interpreted as induced dipoles. The expressions in Eq. (17) for the conductive disk are recovered in the limit \(\epsilon_\text{in}/\epsilon_\text{out} \to \infty\). Note that the resulting electric field inside a dielectric disk is uniform \(\mathbf{E}^{\text{in}} = -\nabla U^{\text{in}} = 2E_0/\epsilon_\text{out}/(\epsilon_\text{in} + \epsilon_\text{out})\hat{x}\).

Similarly, external stress induces multipoles in elastic systems. For example, consider a circular hole of radius \(R\) in an infinite elastic matrix. Under external stress \(\sigma^{\text{ext}}_{xx} = -\sigma_0\), the resultant Airy stress function is obtained by solving
the governing Eq. \([6]\) with \(\rho = 0\) with the traction free boundary condition \((\sigma_{rr} = \sigma_{r\varphi} = 0)\) at the circumference of the hole. The Airy stress function outside the hole \((r > R)\) in polar coordinates is given by \([39]\)

\[
\chi^\text{out}(r, \varphi) = -\frac{\sigma_0 r^2}{4} (1 - \cos 2\varphi) + \frac{\sigma_0 R^2}{2} \ln r - \frac{\sigma_0 R^2}{2} \cos 2\varphi + \frac{\sigma_0 R^4}{4r^2} \cos 2\varphi.
\]  

(19)

The above equation for the Airy stress function reveals that the external stress induces a quadrupole \(P\) (Eq. \([13]\)), a quadrupole \(Q\) (Eq. \([12]\)) and a higher order multipole (Eq. \([15]\)) at the center of the hole (see Fig. \([1]\)). Note that unlike in electrostatics, dipoles are not induced in elasticity. Thus is because isolated disclinations (monopoles) and dislocations (dipoles) are formed by insertion or removal of material and they are thus topological defects \([35]\). On the other hand, elastic quadrupoles \(Q\) and \(P\) can be obtained by local material rearrangement and can thus be induced by external loads \([40]\).

The above analysis can be generalized to the case with a circular inclusion of radius \(R\) made from material with Young’s modulus \(E_{in}\) and Poisson’s ratio \(\nu_{in}\) that is embedded in an infinite elastic matrix made from material with Young’s modulus \(E_{out}\) and Poisson’s ratio \(\nu_{out}\). Under uniaxial compressive stress \(\sigma^\text{ext} = -\sigma_0\), the Airy stress function corresponding to the external stress is \(\chi^\text{ext} = -\sigma_0 y^2/2 = -\sigma_0 r^2 (1 - \cos 2\varphi)/4\). Since the Airy stress function due to external stress contains both the axisymmetric and the \(\cos 2\varphi\) term, the Airy stress function due to induced multipoles should have the same angular dependence. Furthermore stresses should remain finite at the center of the inclusion \((r = 0)\) and also far away from the inclusion \((r \to \infty)\). The total Airy stress function \(\chi^\text{in}(r, \varphi)\) inside \((r < R)\) and \(\chi^\text{out}(r, \varphi)\) outside \((r > R)\) the inclusion can thus be written in the following form

\[
\chi^\text{in}(r, \varphi) = -\frac{\sigma_0 r^2}{4} (1 - \cos 2\varphi) + c_0 r^2 + a_2 r^2 \cos 2\varphi + c_2 \frac{R^2}{r^2} \cos 2\varphi,
\]  

(20a)

\[
\chi^\text{out}(r, \varphi) = -\frac{\sigma_0 r^2}{4} (1 - \cos 2\varphi) + A_0 R^2 \ln \left(\frac{r}{R}\right) + C_2 R^2 \cos 2\varphi + A_2 R^4 r^{-2} \cos 2\varphi.
\]  

(20b)

The last three terms in Eq. \((20b)\) correspond to induced quadrupoles \((P, Q)\) and a higher order multipole at the center of the inclusion, similar to induced multipoles at the center of the hole in Eq. \((19)\). The last three terms in Eq. \((20a)\) can also be interpreted as induced multipoles that are located far away from the inclusion. The unknown coefficients are determined from the boundary conditions, which require that tractions \((\sigma_{rr} \text{ and } \sigma_{r\varphi})\) and displacements \((u_r \text{ and } u_\varphi)\) are continuous at the circumference of the inclusion \((r = R)\). Stresses corresponding to the Airy stress function \(\chi(r, \varphi)\) can be calculated as \(\sigma_{rr} = r^{-1}(\partial \chi / \partial r) + r^{-2}(\partial^2 \chi / \partial r^2)\), \(\sigma_{r\varphi} = \partial \chi / \partial \varphi\), and \(\sigma_{\varphi\varphi} = -\partial(r^{-1} \partial \chi / \partial r) / \partial r\). Table II summarizes the stresses corresponding to different terms in the Michell solution \([33]\). The boundary conditions for

\[
\begin{array}{ccccccc}
\chi & \sigma_{rr} & \sigma_{r\varphi} & \sigma_{\varphi\varphi} & 2\mu \left(\frac{u_r}{u_\varphi}\right) \\
\hline
r^2 & 2 & 0 & 2 & r \left(\frac{\kappa - 1}{\kappa + 1}\right) \\
\ln r & r^{-2} & 0 & -r^{-2} & r^{-1} \left(\frac{1}{\kappa + 1}\right) \\
r^{\nu+2} \cos n\varphi & -(n+1)(n-2) r^n \cos n\varphi & n(n+1) r^n \sin n\varphi & (n+1)(n+2) r^n \cos n\varphi & r^{\nu+1} \left(\frac{\kappa - n - 1}{\kappa + 1}\right) \cos n\varphi \\
r^{\nu+2} \sin n\varphi & -(n+1)(n-2) r^n \sin n\varphi & -n(n+1) r^n \cos n\varphi & (n+1)(n+2) r^n \sin n\varphi & r^{\nu+1} \left(\frac{\kappa - n - 1}{\kappa + 1}\right) \sin n\varphi \\
r^{-n+2} \cos n\varphi & -(n+2)(n-1) r^{-n} \cos n\varphi & -n(n-1) r^{-n} \sin n\varphi & (n-1)(n-2) r^{-n} \cos n\varphi & r^{-n+1} \left(\frac{\kappa - n + 1}{\kappa - n - 1}\right) \cos n\varphi \\
r^{-n+2} \sin n\varphi & -(n+2)(n-1) r^{-n} \sin n\varphi & n(n-1) r^{-n} \cos n\varphi & (n-1)(n-2) r^{-n} \sin n\varphi & r^{-n+1} \left(\frac{\kappa - n + 1}{\kappa - n - 1}\right) \sin n\varphi \\
r^n \cos n\varphi & -(n-1) r^{n-2} \cos n\varphi & n(n-1) r^{n-2} \sin n\varphi & n(n-1) r^{n-2} \cos n\varphi & r^{n-1} \left(\frac{\kappa - n}{\kappa - n + 1}\right) \cos n\varphi \\
r^n \sin n\varphi & -(n-1) r^{n-2} \sin n\varphi & n(n-1) r^{n-2} \sin n\varphi & n(n-1) r^{n-2} \sin n\varphi & r^{n-1} \left(\frac{\kappa - n}{\kappa - n + 1}\right) \sin n\varphi \\
r^{-n} \cos n\varphi & -(n+1) r^{-n-2} \cos n\varphi & n(n+1) r^{-n-2} \sin n\varphi & n(n+1) r^{-n-2} \cos n\varphi & r^{-n-1} \left(\frac{\kappa - n}{\kappa - n + 1}\right) \cos n\varphi \\
r^{-n} \sin n\varphi & -(n+1) r^{-n-2} \sin n\varphi & n(n+1) r^{-n-2} \cos n\varphi & n(n+1) r^{-n-2} \sin n\varphi & r^{-n-1} \left(\frac{\kappa - n}{\kappa - n + 1}\right) \sin n\varphi \\
\end{array}
\]
tractions \((\sigma_{rr} \text{ and } \sigma_{r\varphi})\) at the circumference of inclusion are thus written as

\[
\begin{align*}
\frac{-\sigma_0}{2} + 2\sigma_0 - \left(\frac{\sigma_0}{2} + 2a_2\right) \cos 2\varphi &= -\frac{\sigma_0}{2} + A_0 - \left(\frac{\sigma_0}{2} + 4C + 6A_2\right) \cos 2\varphi, \\
\left(\frac{-\sigma_0}{2} + 2a_2 + 6c_2\right) \sin 2\varphi &= \left(\frac{-\sigma_0}{2} - 2C - 6A_2\right) \sin 2\varphi.
\end{align*}
\]

(21a)

(21b)

In order to obtain displacements, we first calculate strains as 
\(\varepsilon_{rr} = ((k + 1)\sigma_{rr} - (3 - \kappa)\sigma_{r\varphi})/(8\mu), \varepsilon_{r\varphi} = \sigma_{r\varphi}/(2\mu)\)
and 
\(\varepsilon_{\varphi\varphi} = ((k + 1)\sigma_{\varphi\varphi} - (3 - \kappa)\sigma_{rr})/(8\mu)\), where \(\mu = E/(1 + \nu)\) is the shear modulus and we introduced the Kolosov’s constants \(\kappa = (3 - \nu)/(1 + \nu)\) for the plane stress and \(\kappa = 3 - 4\nu\) for the plane strain conditions [32]. Displacements \(u_r\) and \(u_\varphi\) are then obtained via integration of strains. Table I summarizes the displacements corresponding to different terms in the Michell solution [33]. The boundary conditions for displacements \((u_r \text{ and } u_\varphi)\) at the circumference of inclusion are thus written as

\[
\begin{align*}
\left(\frac{-\sigma_0}{4} + c_0\right) \frac{R(k - 1)}{2\mu_{in}} &+ \left(\frac{-\sigma_0}{2} - 2a_2 + c_2(k - 3)\right) \frac{R \cos 2\varphi}{2\mu_{in}} = \\
- \frac{R}{2\mu_{out}} \left(\frac{1}{4}\sigma_0(k - 1) + A_0\right) + \left(\frac{-\sigma_0}{2} + C_2(k - 1) + 2A_2\right) \frac{R \cos 2\varphi}{2\mu_{out}},
\end{align*}
\]

(22a)

(22b)

The boundary conditions in equations (21) and (22) have to be satisfied at every point \((\varphi)\) on the circumference of the inclusion. Thus the coefficients of Fourier components have to match on both sides in these equations, which enables us to rewrite the boundary conditions as a matrix equation

\[
\begin{pmatrix}
1 & 0 & 0 & -2 & 0 & 0 \\
0 & -6 & -4 & 0 & 2 & 0 \\
0 & -6 & -2 & 0 & -2 & -6 \\
-\frac{R}{2\mu_{out}} & 0 & 0 & -\frac{R(k - 1)}{2\mu_{in}} & 0 & 0 \\
0 & \frac{R}{\mu_{out}} & \frac{R(k + 1)}{2\mu_{out}} & 0 & \frac{R}{\mu_{in}} & \frac{R(k - 3)}{2\mu_{in}} \\
0 & \frac{R}{\mu_{out}} & \frac{R(k + 1)}{2\mu_{out}} & 0 & -\frac{R}{\mu_{in}} & \frac{R(k - 3)}{2\mu_{in}}
\end{pmatrix}
\begin{pmatrix}
A_0 \\
A_2 \\
C_2 \\
\frac{\sigma_0 R}{8} - \frac{(k - 1)}{\mu_{out}} - \frac{(k - 1)}{\mu_{in}} \\
\frac{\sigma_0 R}{4} \left(\frac{1}{\mu_{out}} - \frac{1}{\mu_{in}}\right) \\
\frac{\sigma_0 R}{4} \left(\frac{1}{\mu_{out}} + \frac{1}{\mu_{in}}\right)
\end{pmatrix}
= 0.
\]

(23)

By solving the above set of equations we find that the Airy stress functions \(\chi_{in}(r, \varphi)\) inside \((r < R)\) and \(\chi_{out}(r, \varphi)\) outside \((r > R)\) the inclusion are given by

\[
\begin{align*}
\chi_{in}(r, \varphi) &= -\frac{\sigma_0 r^2}{4} (1 - \cos 2\varphi) + \frac{\mu_{out}(k - 1) - \mu_{in}(k - 1)}{4(\mu_{out}(k - 1) + 2\mu_{in}(k - 1))} \sigma_0 r^2 - \frac{\mu_{out} - \mu_{in}}{4(\mu_{out} + \mu_{in})} \frac{\sigma_0 R^2 \cos 2\varphi}{2(\mu_{out} + \mu_{in})}, \\
\chi_{out}(r, \varphi) &= -\frac{\sigma_0 r^2}{4} (1 - \cos 2\varphi) + \frac{\mu_{out}(k - 1) - \mu_{in}(k - 1)}{2(\mu_{out}(k - 1) + 2\mu_{in}(k - 1))} \sigma_0 R^2 \ln r - \frac{\mu_{out} - \mu_{in}}{2(\mu_{out} + \mu_{in})} \frac{\sigma_0 R^2 \cos 2\varphi}{2(\mu_{out} + \mu_{in})k_{out}}, \\
&+ \frac{\mu_{out} - \mu_{in}}{4(\mu_{out} + \mu_{in})k_{out}} \sigma_0 R^4 r^2 \cos 2\varphi.
\end{align*}
\]

(24a)

(24b)

In the above equations (24) for the Airy stress functions the second and third terms can again be interpreted as induced quadrupoles \(P\) and \(Q\). The expressions in Eq. (19) for the hole are recovered in the limit \(\mu_{in} \to 0\). Note that similar to the Eshelby inclusions in 3D [31], the stress field inside the inclusion in 2D is uniform and is given by

\[
\begin{align*}
\sigma_{xx}^{in} &= -\sigma_0 \frac{\mu_{in}(1 + k_{in})(\mu_{out} k_{in} + \mu_{in}(2 + k_{out}))}{2(\mu_{out}(k_{in} - 1) + 2\mu_{in})(\mu_{out} + \mu_{in})k_{out}}, \\
\sigma_{yy}^{in} &= \sigma_0 \frac{\mu_{in}(1 + k_{out})(\mu_{out}(k_{in} - 2) - \mu_{in}(k_{out} - 2))}{2(\mu_{out}(k_{in} - 1) + 2\mu_{in})(\mu_{out} + \mu_{in})k_{out}}, \\
\sigma_{xy}^{in} &= 0.
\end{align*}
\]

(25)

By comparing the above analyses in elasticity and electrostatics, we conclude that holes and inclusions in elasticity are analogous to perfect conductors and dielectrics in electrostatics, respectively.

The problem of induction becomes much more involved when multiple dielectric objects are considered in electrostatics or multiple inclusions in elasticity. This is because dielectric objects and inclusions interact with each other via induced electric fields and stress fields, respectively. In the next section, we describe how such interactions can be systematically taken into account in elasticity, which enabled us to calculate the magnitudes of induced multiples in the presence of external load.
III. ELASTIC MULTIPOLe METHOD

Building on the concepts described above, we developed a method for calculating the linear deformation of circular inclusions and holes embedded in an infinite elastic matrix under external stress. External stress induces elastic multipoles at the centers of inclusions and holes, and their amplitudes are obtained from boundary conditions between different materials (continuity of tractions and displacements). In the following Section III.A, we describe the method for the general case where all circular inclusions are of different sizes and they have different material properties (holes correspond to zero shear modulus). Note that our method applies to the deformation of holes and inclusions embedded in thin plates (plane stress) as well as for cylindrical inclusions and holes embedded in an infinitely thick elastic matrix (plane strain), by appropriately setting the values of Kolosov’s constant. In Section III.B, we compare the results of our method to the finite element simulations and in Section III.C, they are compared to experiments.

A. Method

Let us consider a 2D infinite elastic matrix with the Young’s modulus $E_0$ and Poisson’s ratio $\nu_0$. Embedded in the matrix are $N$ circular inclusions with radii $R_i$ centered at positions $(x_i, y_i)$ with Young’s moduli $E_i$ and Poisson’s ratios $\nu_i$, where $i \in \{1, \ldots, N\}$. Holes are described with the zero Young’s modulus ($E_i = 0$). External stress, represented with the Airy stress function

$$\chi_{\text{ext}}(x, y) = \frac{1}{2} \sigma_{xx}^\text{ext} x^2 + \frac{1}{2} \sigma_{yy}^\text{ext} y^2 - \sigma_{xy}^\text{ext} xy,$$

(26)

induces quadrupoles $Q$, $P$ and higher order elastic multipoles at the centers of inclusions, as was discussed in Sec. II.F. Thus the Airy stress function outside the $i^{th}$ inclusion due to the induced multipoles can be expanded as

$$\chi_{\text{out}}^i(r_i, \varphi_i | a_{\text{in}}^{(i)}) = A_0^{(i)} R_i^2 \ln \left( \frac{r_i}{R_i} \right) + \sum_{n=1}^{\infty} R_i^2 \left[ A_n^{(i)} \left( \frac{r_i}{R_i} \right)^{-n} \cos(n\varphi_i) + B_n^{(i)} \left( \frac{r_i}{R_i} \right)^{-n} \sin(n\varphi_i) \right]$$

$$+ \sum_{n=2}^{\infty} R_i^2 \left[ C_n^{(i)} \left( \frac{r_i}{R_i} \right)^{-n+2} \cos(n\varphi_i) + D_n^{(i)} \left( \frac{r_i}{R_i} \right)^{-n+2} \sin(n\varphi_i) \right],$$

(27)

where the origin of polar coordinates $(r_i, \varphi_i)$ is at the center of the $i^{th}$ inclusion and we introduced the set of amplitudes of induced multipoles $a_{\text{in}}^{(i)} = \{ A_0^{(i)}, A_1^{(i)}, \ldots, B_1^{(i)}, B_2^{(i)}, \ldots, C_2^{(i)}, C_3^{(i)}, \ldots, D_2^{(i)}, D_3^{(i)}, \ldots \}$. The total Airy stress function outside all inclusions can then be written as

$$\chi_{\text{out}}(x, y | a_{\text{in}}) = \chi_{\text{ext}}(x, y) + \sum_{j=1}^{N} \chi_{\text{out}}^j(r_j(x, y), \varphi_j(x, y) | a_{\text{in}}^{(j)}),$$

(28)

where the first term is due to external stress and the summation describes contributions due to induced multipoles at the centers of inclusions. Polar coordinates centered at the origin of the $i^{th}$ inclusion are obtained as $r_i(x, y) = \sqrt{(x - x_i)^2 + (y - y_i)^2}$ and $\varphi_i(x, y) = \arctan((y - y_i)/(x - x_i))$. The set of amplitudes of induced multipoles for all inclusions is defined as $a_{\text{out}} = \{ a_1^{(i)}, \ldots, a_3^{(i)} \}$. Similarly, we expand the induced Airy stress function inside the $i^{th}$ inclusion as

$$\chi_{\text{in}}(r_i, \varphi_i | a_{\text{in}}^{(i)}) = c_0^{(i)} r_i^2 + \sum_{n=2}^{\infty} R_i^2 \left[ c_n^{(i)} \left( \frac{r_i}{R_i} \right)^n \cos(n\varphi_i) + d_n^{(i)} \left( \frac{r_i}{R_i} \right)^n \sin(n\varphi_i) \right]$$

$$+ \sum_{n=1}^{\infty} R_i^2 \left[ e_n^{(i)} \left( \frac{r_i}{R_i} \right)^{n+2} \cos(n\varphi_i) + f_n^{(i)} \left( \frac{r_i}{R_i} \right)^{n+2} \sin(n\varphi_i) \right],$$

(29)

where we keep only terms that generate finite stresses at the center of inclusion and we omitted linear terms ($r_i \cos \varphi_i$, $r_i \sin \varphi_i$) that correspond to zero stresses. The set of amplitudes of induced multipoles is represented as $a_{\text{in}}^{(i)} = \{ a_2^{(i)}, a_3^{(i)}, b_2^{(i)}, b_3^{(i)}, c_0^{(i)}, c_1^{(i)}, \ldots, d_2^{(i)}, d_3^{(i)}, \ldots \}$. The total Airy stress function inside the $i^{th}$ inclusion is thus

$$\chi_{\text{in}, i}(x, y | a_{\text{in}}^{(i)}) = \chi_{\text{ext}}(x, y) + \chi_{\text{in}}^i(r_i(x, y), \varphi_i(x, y) | a_{\text{in}}^{(i)}),$$

(30)

where the first term is due to external stress and the second term is due to induced multipoles.
χ to consider the contributions due to induced multipoles stresses and displacements corresponding to each term in the Michell solution [33]. However, it is not straightforward coordinates \((r_i, \varphi_i)\) at the center of the \(j^{th}\) inclusion to the polar coordinates \((r_j, \varphi_j)\) at the center of the \(i^{th}\) inclusion, we use the cosine and sine rules on the red triangle.

The amplitudes of induced multipoles \(a_{\text{out}}^{(i)}\) and \(a_{\text{in}}^{(i)}\) are obtained by satisfying the boundary conditions that tractions and displacements are continuous across the circumference of each inclusion

\[
\sigma_{\text{tot,rr}}^{\text{in}}(r_i, x, y) = R_i, \varphi_i(x, y) a_{\text{in}}^{(i)}(31a) \\
\sigma_{\text{tot,rr}}^{\text{out}}(r_i, x, y) = R_i, \varphi_i(x, y) a_{\text{out}}^{(i)}, \\
\sigma_{\text{tot,\varphi\varphi}}^{\text{in}}(r_i, x, y) = R_i, \varphi_i(x, y) a_{\text{in}}^{(i)}(31b) \\
\sigma_{\text{tot,\varphi\varphi}}^{\text{out}}(r_i, x, y) = R_i, \varphi_i(x, y) a_{\text{out}}^{(i)}, \\
u_{\text{tot,rr}}^{\text{in}}(r_i, x, y) = R_i, \varphi_i(x, y) a_{\text{in}}^{(i)}(31c) \\
u_{\text{tot,rr}}^{\text{out}}(r_i, x, y) = R_i, \varphi_i(x, y) a_{\text{out}}^{(i)},
\]

where stresses and displacements are obtained from the total Airy stress functions \(\chi_{\text{in}}^{\text{tot,i}}(x, y) a_{\text{in}}^{(i)}\) inside the \(i^{th}\) inclusion (see Eq. (30)) and \(\chi_{\text{out}}^{\text{tot}}(x, y) a_{\text{out}}\) outside all inclusions (see Eq. (28)). In boundary conditions for the \(i^{th}\) inclusion in the above Eq. (31), we can easily take into account contributions due to external stresses and due to induced multipoles \(a_{\text{in}}^{(i)}\) and \(a_{\text{out}}^{(i)}\) in that inclusion. This can be done with the help of Table I which shows the values of stresses and displacements corresponding to each term in the Michell solution [33]. However, it is not straightforward to consider the contributions due to induced multipoles \(a_{\text{out}}^{(j)}\) for other inclusions \((j \neq i)\), because the corresponding Airy stress functions \(\chi_{\text{out}}^{\text{tot}}(r_j, \varphi_j) a_{\text{in}}^{(j)}\) in Eq. (27) are written in the polar coordinates centered around \((x_j, y_j)\). Polar coordinates \((r_j, \varphi_j)\) centered around the \(j^{th}\) inclusion can be expressed in terms of polar coordinates centered around the \(i^{th}\) inclusion as \(r_j(r_i, \varphi_i) = (r_i^2 + a_{ij}^2 - 2r_i a_{ij} \cos(\varphi_i - \theta_{ij}))^{1/2}\), and \(\varphi_j(r_i, \varphi_i) = \pi + \theta_{ij} - \arcsin(r_i \sin(\varphi_i - \theta_{ij})) / r_j\), where \(a_{ij}\) is the distance between the centers of the \(i^{th}\) and \(j^{th}\) inclusion and \(\theta_{ij}\) is the angle between the line joining the centers of inclusions and the x-axis as shown in Fig. 3. The Airy stress function due to induced multipoles centered around the \(j^{th}\) inclusion can be expanded in Taylor series around the center of \(i^{th}\) inclusion as [28]

\[
\chi_{\text{out}}^{\text{tot}}(r_j, \varphi_j) a_{\text{out}}^{(j)} = \sum_{n=2}^{\infty} R_j^{n+2} a_{ij}^{n+2} \left[ \cos(n \varphi_i) f_r^n \left( R_j / a_{ij}, \theta_{ij} | a_{\text{out}}^{(j)} \right) + \sin(n \varphi_i) f_r^n \left( R_j / a_{ij}, \theta_{ij} | a_{\text{out}}^{(j)} \right) \sin(n \varphi_i) g_r^n \left( R_j / a_{ij}, \theta_{ij} | a_{\text{out}}^{(j)} \right) \right],
\]

where we omitted linear terms \((r_i \cos \varphi_i, r_i \sin \varphi_i)\) that correspond to zero stresses and we introduced new functions

\[
f_r^n \left( R_j / a_{ij}, \theta_{ij} | a_{\text{out}}^{(j)} \right) = \sum_{m=0}^{\infty} \left[ A_m^{(j)} A_r^n(\theta_{ij}) + B_m^{(j)} B_n^r(\theta_{ij}) \right] + R_{m+2}^{j} a_{ij}^{m+2} \left[ C_m^{(j)} C_n^m(\theta_{ij}) + D_m^{(j)} D_n^m(\theta_{ij}) \right], \tag{33a}
\]

\[
f_r^n \left( R_j / a_{ij}, \theta_{ij} | a_{\text{out}}^{(j)} \right) = \sum_{m=0}^{\infty} \left[ A_m^{(j)} B_r^n(\theta_{ij}) - B_m^{(j)} A_n^r(\theta_{ij}) \right] + R_{m+2}^{j} a_{ij}^{m+2} \left[ C_m^{(j)} D_n^m(\theta_{ij}) - D_m^{(j)} C_n^m(\theta_{ij}) \right], \tag{33b}
\]
Next we calculate stresses and displacement at the circumference of the inclusion $R_j$. In Eq. (32) we set $B_0^{(j)} = C_0^{(j)} = D_0^{(j)} = C_1^{(j)} = D_1^{(j)} = 0$ and we introduced coefficients $A_n^m(\theta_{ij}), B_n^m(\theta_{ij}), C_n^m(\theta_{ij}), D_n^m(\theta_{ij}), C_1^m(\theta_{ij}), D_1^m(\theta_{ij})$, and $E_n^m(\theta_{ij})$ that are summarized in Table III.

We next calculate stresses and displacement at the circumference of the $i$th inclusion by using expressions for the Airy stress functions due to external stresses in Eq. (26), due to induced multipoles for the $i$th inclusion in Eqs. (27) and (29), and due to induced multipoles for the $j$th inclusion ($j \neq i$) in Eq. (32). With the help of Table II we obtain

$$
\sigma_{i,j}^{(0)}(r_i, \varphi_i | a_{i,j}^{(0)}) = \frac{1}{2} \left( \sigma_{xx}^{ext} + \sigma_{yy}^{ext} \right) + \frac{1}{2} \left( \sigma_{xx}^{ext} - \sigma_{yy}^{ext} \right) \sin(2 \varphi_i) + \sigma_{xy}^{ext} \cos(2 \varphi_i) + 2 \frac{c_0}{a_{i,j}^{(0)}}
$$

$$
\sigma_{i,j}^{(n)}(r_i, \varphi_i | a_{i,j}^{(n)}) = \frac{1}{2} \left( \sigma_{xx}^{ext} + \sigma_{yy}^{ext} \right) + \frac{1}{2} \left( \sigma_{xx}^{ext} - \sigma_{yy}^{ext} \right) \sin(2 \varphi_i) + \sigma_{xy}^{ext} \cos(2 \varphi_i) + A^{(i)} + \sum_{n=1}^{\infty} \left[ n(n-1) \left( a_n^{(i)} \cos(n \varphi_i) + b_n^{(i)} \sin(n \varphi_i) \right) + (n+1)(n-2) \left( c_n^{(i)} \cos(n \varphi_i) + d_n^{(i)} \sin(n \varphi_i) \right) \right],
$$

$$
\sigma_{i,j}^{(0)}(r_i, \varphi_i | a_{i,j}^{(0)}) = \frac{1}{2} \left( \sigma_{xx}^{ext} + \sigma_{yy}^{ext} \right) \sin(2 \varphi_i) + \sigma_{xy}^{ext} \cos(2 \varphi_i) + 2 \frac{c_0}{a_{i,j}^{(0)}}
$$

$$
\sigma_{i,j}^{(n)}(r_i, \varphi_i | a_{i,j}^{(n)}) = \frac{1}{2} \left( \sigma_{xx}^{ext} + \sigma_{yy}^{ext} \right) \sin(2 \varphi_i) + \sigma_{xy}^{ext} \cos(2 \varphi_i) + A^{(i)} + \sum_{n=1}^{\infty} \left[ n(n+1) \left( a_n^{(i)} \cos(n \varphi_i) + b_n^{(i)} \cos(n \varphi_i) \right) + (n+1)(n-2) \left( c_n^{(i)} \sin(n \varphi_i) \right) \right],
$$

$$
\sigma_{i,j}^{(0)}(r_i, \varphi_i | a_{i,j}^{(0)}) = \frac{1}{2} \left( \sigma_{xx}^{ext} - \sigma_{yy}^{ext} \right) \sin(2 \varphi_i) + \sigma_{xy}^{ext} \cos(2 \varphi_i) + 2 \frac{c_0}{a_{i,j}^{(0)}}
$$
\[
\frac{2\mu_i}{R_i} u_{\text{tot},r}^{\text{in}}(r_i = R_i, \varphi_i a_{\text{in}}^{(i)}) = \frac{1}{4} (\sigma_{xx}^{\text{ext}} + \sigma_{yy}^{\text{ext}})(\kappa_i - 1) + \frac{1}{2} (\sigma_{xx}^{\text{ext}} - \sigma_{yy}^{\text{ext}}) \cos(2\varphi_i) + \sigma_{xy}^{\text{ext}} \sin(2\varphi_i) + c_0^{(i)} (\kappa_i - 1)
- \sum_{n=1}^{\infty} \left[ n \left( a_n^{(i)} \cos(n\varphi_i) + b_n^{(i)} \sin(n\varphi_i) \right) + (n + 1 - \kappa_i) \left( c_n^{(i)} \cos(n\varphi_i) + d_n^{(i)} \sin(n\varphi_i) \right) \right],
\]

(34e)

\[
\frac{2\mu_0}{R_i} u_{\text{tot},r}^{\text{out}}(r_i = R_i, \varphi_i a_{\text{out}}^{(i)}) = \frac{1}{4} (\sigma_{xx}^{\text{ext}} + \sigma_{yy}^{\text{ext}})(\kappa_i - 1) + \frac{1}{2} (\sigma_{xx}^{\text{ext}} - \sigma_{yy}^{\text{ext}}) \cos(2\varphi_i) + \sigma_{xy}^{\text{ext}} \sin(2\varphi_i) - A_0^{(i)}
+ \sum_{n=1}^{\infty} \left[ n \left( A_n^{(i)} \cos(n\varphi_i) + B_n^{(i)} \sin(n\varphi_i) \right) + (\kappa_0 + n - 1) \left( C_n^{(i)} \cos(n\varphi_i) + D_n^{(i)} \sin(n\varphi_i) \right) \right]
- \sum_{j \neq i} \sum_{n=2}^{\infty} \frac{R_j^2 R_i^{-2}}{a_{ij}^{n}} n \left[ \cos(n\varphi_i) f_c^{(n)} \left( R_j/a_{ij}, \theta_{ij} | a_{\text{in}}^{(j)} \right) + \sin(n\varphi_i) f_s^{(n)} \left( R_j/a_{ij}, \theta_{ij} | a_{\text{out}}^{(j)} \right) \right]
+ \sum_{j \neq i} \sum_{n=0}^{\infty} \frac{R_j^2 R_i^{-2}}{a_{ij}^{n+2}} (\kappa_0 - n - 1) \left[ \cos(n\varphi_i) g_c^{(n)} \left( R_j/a_{ij}, \theta_{ij} | a_{\text{in}}^{(j)} \right) + \sin(n\varphi_i) g_s^{(n)} \left( R_j/a_{ij}, \theta_{ij} | a_{\text{out}}^{(j)} \right) \right],
\]

(34f)

\[
\frac{2\mu_i}{R_i} u_{\text{tot},r}^{\text{in}}(r_i = R_i, \varphi_i a_{\text{in}}^{(i)}) = \frac{1}{2} (\sigma_{xx}^{\text{ext}} - \sigma_{yy}^{\text{ext}}) \sin(2\varphi_i) + \sigma_{xy}^{\text{ext}} \cos(2\varphi_i)
+ \sum_{n=1}^{\infty} \left[ n \left( a_n^{(i)} \sin(n\varphi_i) - b_n^{(i)} \cos(n\varphi_i) \right) + (\kappa_i + n + 1) \left( c_n^{(i)} \sin(n\varphi_i) - d_n^{(i)} \cos(n\varphi_i) \right) \right]
\]

(34g)

\[
\frac{2\mu_0}{R_i} u_{\text{tot},r}^{\text{out}}(r_i = R_i, \varphi_i a_{\text{out}}^{(i)}) = \frac{1}{2} (\sigma_{xx}^{\text{ext}} - \sigma_{yy}^{\text{ext}}) \sin(2\varphi_i) + \sigma_{xy}^{\text{ext}} \cos(2\varphi_i)
+ \sum_{n=1}^{\infty} \left[ n \left( A_n^{(i)} \sin(n\varphi_i) - B_n^{(i)} \cos(n\varphi_i) \right) - (\kappa_0 - n + 1) \left( C_n^{(i)} \sin(n\varphi_i) - D_n^{(i)} \cos(n\varphi_i) \right) \right]
+ \sum_{j \neq i} \sum_{n=2}^{\infty} \frac{R_j^2 R_i^{-2}}{a_{ij}^{n}} n \left[ \cos(n\varphi_i) f_c^{(n)} \left( R_j/a_{ij}, \theta_{ij} | a_{\text{in}}^{(j)} \right) + \cos(n\varphi_i) f_s^{(n)} \left( R_j/a_{ij}, \theta_{ij} | a_{\text{out}}^{(j)} \right) \right]
+ \sum_{j \neq i} \sum_{n=0}^{\infty} \frac{R_j^2 R_i^{-2}}{a_{ij}^{n+2}} (\kappa_0 + n + 1) \left[ \sin(n\varphi_i) g_c^{(n)} \left( R_j/a_{ij}, \theta_{ij} | a_{\text{in}}^{(j)} \right) + \cos(n\varphi_i) g_s^{(n)} \left( R_j/a_{ij}, \theta_{ij} | a_{\text{out}}^{(j)} \right) \right].
\]

(34h)

The boundary conditions in Eq. (31) have to be satisfied at every point (\varphi_i) on the circumference of the \ith inclusion. Thus the coefficients of Fourier modes (1, \cos(n\varphi_i), \sin(n\varphi_i)) have to match in the expansions of tractions and displacements in Eq. (33), similar to what was done for a single inclusion in Sec. II F. This enables us to construct a matrix equation for the set of induced multipoles \{a_{\text{in}}^{(i)}, a_{\text{out}}^{(i)}\} in the form (see also Fig. 4)

\[
\begin{pmatrix}
M_{\text{in}}^{\text{disp},ij} & M_{\text{in}}^{\text{disp},in} \\
M_{\text{out}}^{\text{disp},ij} & M_{\text{out}}^{\text{disp},in}
\end{pmatrix}
\begin{pmatrix}
a_{\text{in}}^{(j)} \\
a_{\text{out}}^{(j)}
\end{pmatrix}
= \begin{pmatrix}
0 \\
b_{\text{disp}}^{(i)}
\end{pmatrix},
\]

(35)

where the summation over inclusions \(j\) is implied. The top and bottom rows of matrix \(M\) in the above equation are obtained from the boundary conditions in Eq. (31) for tractions (superscript trac) and displacements (superscript disp), respectively. The left and right columns of matrix \(M\) describe the effect of the induced multipoles \(a_{\text{in}}^{(i)}\) and \(a_{\text{out}}^{(i)}\), respectively. The entries in matrices \(M_{\text{in}}^{\text{disp},ij}\) and \(M_{\text{out}}^{\text{disp},ij}\) for the \ith inclusion are numbers that depend on the degrees of induced multipoles. The entries in matrices \(M_{\text{in}}^{\text{disp},ij}\) and \(M_{\text{out}}^{\text{disp},ij}\) for the \ith inclusion depend on the degrees of induced multipoles, the radius of inclusion \(R_i\) and material properties of inclusion \((\mu_i, \kappa_i)\) and elastic matrix \((\mu_0, \kappa_0)\). Matrices \(M_{\text{in}}^{\text{disp},ij}\) and \(M_{\text{out}}^{\text{disp},ij}\) encode interactions between inclusions \(i\) and \(j\). The entries in these matrices depend on the degrees of induced multipoles, the radii \(R_i\) and \(R_j\) of inclusions, the angle \(\theta_{ij}\) and the separation distance \(a_{ij}\) between inclusions (see Fig. 3). In addition to that, the entries in matrix \(M_{\text{out}}^{\text{disp},ij}\) also depend on the material properties of inclusions \((\mu_i, \kappa_i, \mu_j, \kappa_j)\) and elastic matrix \((\mu_0, \kappa_0)\). Note that the other matrices are zero, i.e. \(M_{\text{in}}^{\text{disp}} = M_{\text{in}}^{\text{disp}} = 0\). The entries in vector \(b_{\text{disp}}^{(i)}\) depend on the magnitude of external stresses \((\sigma_{xx}^{\text{ext}}, \sigma_{yy}^{\text{ext}}, \sigma_{xy}^{\text{ext}})\), the degrees of induced multipoles, the radius of inclusion \(R_i\) and material properties of inclusion \((\mu_i, \kappa_i)\) and elastic matrix \((\mu_0, \kappa_0)\). Note that in \(b_{\text{disp}}^{(i)}\) the only nonzero entries are the ones that correspond to Fourier modes 1, \cos(2\varphi_i), and \sin(2\varphi_i).

In order to numerically solve the system of equations for induced multipoles in Eq. (35) we truncate the multipole expansion at degree \(n_{\text{max}}\). For each inclusion \(i\), there are \(4n_{\text{max}} - 1\) unknown amplitudes of multipoles...
The matrix $\mathbf{M}$ is broken down into $4N^2$ blocks, where blocks $\mathbf{M}_{\text{out},ij}$, $\mathbf{M}_{\text{in},ij}$ correspond to the boundary conditions for tractions around the circumference of $i$th inclusion in Eq. (31a, 31b), and $\mathbf{M}_{\text{disp},ij}$, $\mathbf{M}_{\text{in},ij}$, correspond to the boundary conditions for displacements around the circumference of $i$th inclusion in Eq. (31c, 31d). The red boxes mark the blocks with $i \neq j$ that account for the interactions between different inclusions. The effect of external stresses is contained in vectors $\mathbf{b}_{\text{disp}}$. See main text for the detailed description of elements represented in this system of equations.

![Fourier coefficients of the BC equations](image)

![Effect of multipoles](image)

![Effect of external stress](image)

**FIG. 4.** Structure of the system of equations (35) for the amplitudes of induced multipoles $\mathbf{a}^{(i)}$, $\mathbf{a}^{(i)}$ for inclusions $i \in \{1, \ldots, N\}$. The matrix $\mathbf{M}$ is broken down into $4N^2$ blocks, where blocks $\mathbf{M}_{\text{out},ij}$, $\mathbf{M}_{\text{in},ij}$ correspond to the boundary conditions for tractions around the circumference of $i$th inclusion in Eq. (31a, 31b), and $\mathbf{M}_{\text{disp},ij}$, $\mathbf{M}_{\text{in},ij}$, correspond to the boundary conditions for displacements around the circumference of $i$th inclusion in Eq. (31c, 31d). The red boxes mark the blocks with $i \neq j$ that account for the interactions between different inclusions. The effect of external stresses is contained in vectors $\mathbf{b}_{\text{disp}}$. See main text for the detailed description of elements represented in this system of equations.
B. Comparison with finite element simulations

First we tested the elastic multipole method for two circular inclusions embedded in an infinite plate under plane stress condition subjected to uniaxial stress (Fig. 5) and shear stress (Fig. 6). The two inclusions had identical diameters \( d \) and they were centered at \((\pm a/2,0)\). Three different values of separation distance between inclusions were considered: \( a = 2d \), \( a = 1.4d \), and \( a = 1.1d \). The left and right inclusions were chosen to be more flexible \( (E_1/E_0 = 0.25) \) and more stiff \( (E_2/E_0 = 4) \) than the outer matrix with Young’s modulus \( E_0 \), respectively. The values of applied uniaxial stress and shear stress were \( \sigma_{xx}^{\text{ext}}/E_0 = -0.25 \) (Fig. 5) and \( \sigma_{xy}^{\text{ext}}/E_0 = 0.1 \) (Fig. 6), respectively.

Note that such large values of applied loads fall outside the regime of linear elasticity, where our method applies. Such large values of external loads were used to exaggerate deformations because the amplitudes of induced multipoles are proportional to the magnitude of applied loads.

In Figs. 5 and 6 we show contours of deformed inclusions and spatial distributions of stresses for different values of the separation distance \( a \) between inclusions, where results from the elastic multipole method are compared against finite element simulations on a square domain of size \( 400d \times 400d \) (see Appendix A for details). When inclusions are far from each other they interact weakly, which can be seen from the expansion of stresses and displacements in Eq. (34), where terms describing interactions between inclusions \( i \) and \( j \) contain powers of \( R_i/a_{ij} \ll 1 \) and \( R_j/a_{ij} \ll 1 \). This is the case for the separation distance \( a = 2d \), where we find that contours of deformed inclusions have elliptical shapes (see Figs. 5 p and 6 j) and stresses inside inclusions are uniform (see Figs. 5 h and 6 h), which is characteristic for isolated inclusions (see Eq. (25) and [1]). Furthermore, the von Mises stress distribution \( (\sigma_{VM} = (\sigma_{xx} - \sigma_{yy} + \sigma_{zz})/2) \) around the more flexible left inclusion (see Fig. 5 h) is similar to the one for an isolated hole under uniaxial stress (see Fig. 1 h). For the more stiff right inclusion, the locations of maxima and minima in the von Mises stress distribution are reversed (see Fig. 6 h) because the amplitudes of induced multipoles have the opposite sign (see Eq. (24)). Similar patterns in the von Misses stress distribution are observed when the structure is under external shear, but they are rotated by \( 45^\circ \) (see Fig. 6 l). When inclusions are far apart \((a = 2d)\), the contours of deformed inclusions can be accurately described already with multipoles up to degree \( n_{\text{max}} = 2 \) (see Figs. 5 p and 6 p). This degree of multipoles is sufficient because external stresses \( \sigma_{xx}^{\text{ext}} \) and \( \sigma_{yy}^{\text{ext}} \) couple only to the Fourier modes 1, \( \cos 2\phi \), and \( \sin 2\phi \), in the expansion for stresses and displacements in Eq. (34). When inclusions are moved closer together \((a = 1.4d \text{ and } a = 1.1d)\), they start interacting more strongly. As a consequence contours of deformed inclusions become progressively more non-elliptical and higher order of multipoles are needed to accurately describe their shapes (see Figs. 5 d and 6 d). Furthermore, the stress distribution inside the right inclusion becomes nonuniform (see Figs. 5 g, i, j and 6 g, i, j). Note that von Misses stress distributions look similar far from inclusions regardless of the separation distance \( a \), because they are dictated by the lowest order of induced multipoles, i.e. quadrupoles \( Q \) and \( P \) (see Figs. 5 f and 6 f).

In order to determine the proper number for the maximum degree \( n_{\text{max}} \) of induced multipoles we performed a convergence analysis for the spatial distributions of displacements \( u^{(n_{\text{max}})}(x, y) \) and von Misses stresses \( \sigma_{ VM}^{(n_{\text{max}})}(x, y) \). Displacements and von Misses stresses were evaluated at \( N_p = 1001 \times 1001 \) points on a square grid of size \( 10d \times 10d \) surrounding the inclusions, i.e. at points \((x, y) = (id/100, jd/100)\), where \( i, j = -500, -499, \ldots, 500 \). The normalized errors for displacements \( \epsilon_{\text{disp}}(n_{\text{max}}) \) and stresses \( \epsilon_{\text{stress}}(n_{\text{max}}) \) were obtained by calculating the relative changes of the spatial distributions of displacements and von Misses stresses, when the maximum degree \( n_{\text{max}} \) of induced multipoles is increased by one. The normalized errors are given by [50]

\[
\epsilon_{\text{disp}}(n_{\text{max}}) = \frac{1}{\sqrt{N_p}} \left[ \sum_{i,j} \left( \frac{u_x^{(n_{\text{max}}+1)}(x_i, y_j) - u_x^{(n_{\text{max}})}(x_i, y_j)}{d} \right)^2 + \left( \frac{u_y^{(n_{\text{max}}+1)}(x_i, y_j) - u_y^{(n_{\text{max}})}(x_i, y_j)}{d} \right)^2 \right]^{1/2},
\]

\[
\epsilon_{\text{stress}}(n_{\text{max}}) = \frac{1}{\sqrt{N_p}} \left[ \sum_{i,j} \left( \sigma_{VM}^{(n_{\text{max}}+1)}(x_i, y_j) - \sigma_{VM}^{(n_{\text{max}})}(x_i, y_j) \right)^2 \right]^{1/2},
\]

where displacements and von Misses stresses are normalized by the diameter \( d \) of inclusions and by the value of the von Misses stress \( \sigma_{ VM}^{ext} \) due to external load, respectively. The normalized errors are plotted in Fig. 7. As the maximum degree \( n_{\text{max}} \) of induced multipoles is increased, the normalized errors for displacements \( \epsilon_{\text{disp}}(n_{\text{max}}) \) and stresses \( \epsilon_{\text{stress}}(n_{\text{max}}) \) decrease exponentially. Because the induced elastic multipoles form the basis for the biharmonic equation, this is akin to the spectral method, which is exponentially convergent when the functions and the shapes of boundaries are smooth [50]. The normalized errors for displacements are lower than the errors for stresses (see Fig. 7), because stresses are related to spatial derivatives of displacements. Note that the normalized errors decrease more slowly, when inclusions are brought close together and their interactions become important (see Fig. 7). This is
FIG. 5. Deformation of an infinite elastic plate with two circular inclusions under uniaxial stress. (a) Schematic image describing the initial undeformed shape of the structure and the direction of applied load $\sigma_{xx}^{\text{ext}} = -0.25 E_0$. The diameters of both inclusions (blue and orange) are $d$ and the separation distance between their centers is $a$. The Young’s moduli of the left and right inclusions are $E_1/E_0 = 0.25$ and $E_2/E_0 = 4$, respectively, where $E_0$ is the Young’s modulus of the outer material. The Poisson’s ratios of the left and right inclusions, and the outer material are $\nu_1 = 0.45$, $\nu_2 = 0.15$ and $\nu_0 = 0.3$, respectively. (b-d) Contours of deformed inclusions for different values of separation distance $a/d = 2, 1.4$ and 1.1. The solid red, yellow and dashed blue lines show the contours obtained with the elastic multipole method for $n_{\text{max}} = 2, 4$ and 8, respectively. Green solid lines represent the contours obtained from finite element simulations. (e-j) von Mises stress ($\sigma_{\text{vM}}$) distributions obtained with (e-g) the elastic multipole method ($n_{\text{max}} = 9$) and (h-j) finite element simulations for different separation distances of inclusions $a/d$. Four marked points A-D are chosen for the quantitative comparison of stresses ($\sigma_{\text{vM}}$) and displacements ($|u|$) between the elastic multipole method (EMP) and finite element simulations (FEM) that are reported in Tables (k-m). The relative percent errors $\epsilon$ between the two methods are calculated as $100 \times (\sigma_{\text{vM}}^{\text{EMP}} - \sigma_{\text{vM}}^{\text{FEM}}) / \sigma_{\text{vM}}^{\text{FEM}}$ and $100 \times (|u|^{\text{EMP}} - |u|^{\text{FEM}}) / |u|^{\text{FEM}}$.

also reflected in the amplitudes $a_{\text{in}}^{(1,2)}$ and $a_{\text{out}}^{(1,2)}$ of induced multipoles, which decrease exponentially with the degree of multipoles and amplitudes decrease more slowly when inclusions are close together (see Fig. 7).

Results of the elastic multipole method were compared against finite element simulations and very good agreement is achieved already for $n_{\text{max}} = 9$ even when inclusions are very close together ($a = 1.1d$, see Figs. 5 and 6). To make the comparison with finite elements more quantitative we compared the values of displacements and stresses at 4 different points: A – at the edge of the left inclusion, B – in between inclusions, C – at the center of the right inclusion, and
FIG. 6. Deformation of an infinite elastic plate with two circular inclusions under shear stress. (a) Schematic image describing the initial undeformed shape of the structure and the direction of applied load $\sigma_{xy}^{ext} = 0.1E_0$, where $E_0$ is the Young's modulus of the outer material. The diameter of both inclusions (blue and orange) is $d$ and the separation distance between their centers is $a$. Material properties are the same as in Fig. 5. (b-d) Contours of deformed inclusions for different values of separation distance $a/d = 2$, $1.4$ and $1.1$. The solid red, yellow and dashed blue lines show the contours obtained with the elastic multipole method for $n_{\text{max}} = 2$, $4$ and $8$, respectively. Green solid lines represent the contours obtained from finite element simulations. (e-j) von Mises stress ($\sigma_{vM}$) distributions obtained with (e-g) the elastic multipole method ($n_{\text{max}} = 9$) and (h-j) finite element simulations for different separation distances of inclusions $a/d$. Four marked points A-D are chosen for the quantitative comparison of stresses ($\sigma_{vM}$) and displacements ($u$) between the elastic multipole method (EMP) and finite element simulations (FEM) that are reported in Tables (k-m). The relative percent errors $\epsilon$ between the two methods are calculated as $100 \times (\sigma_{vM}^{\text{EMP}} - \sigma_{vM}^{\text{FEM}})/\sigma_{vM}^{\text{FEM}}$ and $100 \times (|u|^{\text{EMP}} - |u|^{\text{FEM}})/|u|^{\text{FEM}}$. D – far away from both inclusions (see Figs. 5 and 6). For all 4 points the error increases when inclusions are brought closer together (see Fig. 7). From the 4 different points, we note that the errors are the largest at point B, which is strongly influenced by induced multipoles from both inclusions. The errors for the von Mises stresses at point B are 2.80% and 2.56% for the uniaxial and shear loads, respectively. These errors can be further reduced by increasing the number $n_{\text{max}}$ for the maximum degree of multipoles, e.g. for $n_{\text{max}} = 14$ the errors for von Mises stresses at point B are reduced to 1.20% and 1.09% for the uniaxial and shear loads, respectively.
FIG. 7. Normalized errors and amplitudes of induced multipoles for the structures with two inclusions with diameters $d$ and separation distance $a$ under (a) uniaxial stress (see Fig. 5) and (b) shear stress (see Fig. 6). The normalized errors for displacements $\epsilon_{\text{disp}}(n_{\text{max}})$ (blue lines) and stresses $\epsilon_{\text{stress}}(n_{\text{max}})$ (red lines) are defined in Eq. (36). Absolute values of amplitudes of induced multipoles $\{a^{(1)}_{\text{out}}, a^{(2)}_{\text{out}}\}$ for $n_{\text{max}} = 9$. In (a) amplitudes are normalized, such that $\tilde{a}_n = a_n/\sigma_{\text{ext}}^{\text{ax}}, \tilde{\epsilon}_n = \epsilon_n/\sigma_{\text{ext}}^{\text{ax}}, \tilde{A}_n = A_n/\sigma_{\text{ext}}^{\text{ax}}, \tilde{C}_n = C_n/\sigma_{\text{ext}}^{\text{ax}}$. The dark and light blue colored bars correspond to the positive ($a_n, c_n, A_n, C_n > 0$) and negative ($a_n, c_n, A_n, C_n < 0$) amplitudes for inclusion 1, respectively. Similarly, the red and orange colored bars correspond to the positive and negative amplitudes for inclusion 2, respectively. Note that the amplitudes of multipoles $a_i, c_i, A_i$ and $C_i$ are zero due to the symmetry of the problem. In (b) amplitudes are normalized, such that $\tilde{b}_n = b_n/\sigma_{\text{ext}}^{\text{xy}}, \tilde{d}_n = d_n/\sigma_{\text{ext}}^{\text{xy}}, \tilde{B}_n = B_n/\sigma_{\text{ext}}^{\text{xy}}, \tilde{D}_n = D_n/\sigma_{\text{ext}}^{\text{xy}}$. The dark and light blue colored bars correspond to the positive ($b_n, d_n, B_n, D_n > 0$) and negative ($b_n, d_n, B_n, D_n < 0$) amplitudes for inclusion 1, respectively. Similarly, the red and orange colored bars correspond to the positive and negative amplitudes for inclusion 2, respectively. Note that the amplitudes of multipoles $a_i, c_i, A_i$ and $C_i$ are zero due to the symmetry of the problem.
FIG. 8. Deformation of an infinite elastic plate with ten circular inclusions under general external stress. (a) Schematic image describing the initial undeformed shape of the structure and the direction of applied external loads: $\sigma_{xx}^{\text{ext}}/E_0 = -0.25$, $\sigma_{xy}^{\text{ext}}/E_0 = 0.05$, and $\sigma_{yy}^{\text{ext}}/E_0 = 0.10$, where $E_0$ is the Young’s modulus of the outer material. The radii and material properties (Young’s moduli $E_i$ and Poisson’s ratios $\nu_i$) of inclusions are provided in the table underneath the schematic image. The radii of inclusions are normalized by the radius of the largest inclusion. The Young’s moduli are normalized by the Young’s modulus of the outer material $E_0$. The value of Poisson’s ratio for the outer material is $\nu_0 = 0.3$. (b) Contours of deformed inclusions. The blue dashed lines show the results obtained with the elastic multipole method ($n_{\text{max}} = 6$). Green solid lines correspond to deformed contours obtained with finite element simulation.

To demonstrate the full potential of the elastic multipole method, we also considered the deformation of an infinite plate (plane stress) with $N = 10$ inclusions of different sizes and material properties under general external stress (see Fig. 8). The contours of deformed inclusions obtained with finite element simulations (green solid lines) and the elastic multipole method with $n_{\text{max}} = 6$ (blue dashed lines) agree very well. Note that the results for the elastic multipole method were obtained by solving the linear system of only $N(8n_{\text{max}}-2) = 460$ equations for the amplitudes of induced multipoles described in Eq. (35) (see also Fig. 4), which is significantly smaller than the number of degrees of freedom required for finite element simulations.

C. Comparison with experiments

Finally, we also tested the elastic multipole method against experiments. Experimental samples were prepared by casting Elite Double 32 (Zhermack) elastomers with a measured Young’s modulus $E_0 = 0.97$ MPa and assumed Poisson’s ratio $\nu = 0.49$ [19]. Molds were fabricated from 5 mm thick acrylic plates with laser cut circular holes that were then filled with acrylic cylinders in the assembled molds to produce cylindrical holes in elastomer samples. Approximately 30 min after casting, molds were disassembled and solid samples were placed in a convection oven at $40^\circ$C for 12 hours for further curing. The cylindrical inclusions made from acrylic (Young’s modulus $E = 2.9$ GPa, Poisson’s ratio $\nu = 0.37$ [51]) were inserted into the holes in elastomer samples and fastened by a cyanoacrylate glue where required.

We designed two compressive testing systems (see Fig. 9) to compare the contours of deformed holes/inclusions, and strain fields with predictions from the elastic multipole method. First, we present an experimental system for extracting the contours of deformed holes/inclusions in compressed experimental samples (see Fig. 9a). The apparatus comprises a custom-made loading mechanism and a flatbed photo scanner. Displacement loading is applied incrementally in 0.5 mm steps via $180^\circ$ turns of the M10x1 screw (metric thread 10 mm in diameter and 1 mm pitch) in the mechanism that is controlled with a 3D printed plastic wrench key (see the inset of Fig. 9a). The loading mechanism was placed on an Epson V550 photo scanner to scan the surface of deformed specimens and silicone oil was applied in-between the specimen and the glass surface of the scanner to reduce the friction between them. Scanned
FIG. 9. Experimental systems for displacement controlled compressive tests. (a) A mechanism for compression of rubber specimens (green slab) sits on top of a scanning apparatus, which is used to extract the contours of deformed holes/inclusions. The zoomed-in photo on the left shows a 3D printed plastic wrench key that was used for precise control of screw turns. (b) Setup for extracting strain fields via the digital image correlation (DIC). The surface of specimen was painted with speckle patterns. The specimen was then compressed with steel plates of the testing machine and photos of speckled patterns were used to extract the displacement field on the front surface of the slab. The zoomed-in photo on the right shows the rubber specimen with one hole and one inclusion (indicated with a red dashed circle) mounted between two parallel plates of the testing machine.

images were post-processed with the Image Processing Toolbox in MATLAB 2018b and Corel PHOTO-PAINT X8. First, we digitally removed the dust particles and air-bubbles trapped inside a thin silicone oil film from scanned images. Scanned grayscale images were then tresholded into black and white binary images from which the contours were obtained with MATLAB.

Second, we present a system for capturing the displacement and strain fields in compressed samples via the digital image correlation (DIC) technique (see Fig. 9b). Black and white speckle patterns were spray-painted on the surface of specimens with a slow drying acrylic paint that was used so that the pattern does not dry and harden too fast, which could cause delamination under applied compressive loads. A universal material testing machine Z050 from Zwick was used to apply incremental compressive displacement in steps of 0.2 mm, where we again applied a silicone oil between the steel plates and the elastomer samples to prevent sticking and to reduce friction. At each step, a snapshot of compressed sample was taken with a Nikon D5600 photo camera. These photos were then used to calculate the displacements and strains fields via Ncorr, an open source 2D DIC MATLAB based software.

First, we analyzed uniaxially compressed 100 mm × 100 mm × 25 mm elastomer structures with three different configurations (horizontal, vertical and at 45° angle) of two holes with identical diameters \( d = 8.11 \) mm and their separation distance \( a = 9.50 \) mm (see Fig. 10a-c). Holes were placed near the centers of elastomer structures to minimize the effects of boundaries and elastomer structures were relatively thick 25 mm to prevent the out-of-plane buckling. Contours of deformed holes in compressed experimental samples under external strain \( \varepsilon_{yy}^{ext} = -0.05 \) were compared to the ones obtained from the elastic multipole method and finite element simulations (see Fig. 10d-f). For the elastic multipole method, we used external stress \( \sigma_{yy}^{ext} = E_0 \varepsilon_{yy}^{ext} \) \( (\sigma_{xx} \approx 0 \) due to reduced friction) and the plane stress condition was assumed because experimental samples were free to expand in the out-of-plane direction. Linear finite element simulations were performed for finite size (100 mm × 100 mm) 2D structures with circular holes, where we again assumed the plane stress condition. In finite element simulations samples were compressed by prescribing a uniform displacement in the \( y \) direction on the upper and lower surfaces, while allowing nodes on these surfaces to move freely in the \( x \) direction. (The midpoint of the bottom edge was constrained to prevent rigid body translations in the \( x \) direction.)

The contours of deformed holes obtained in experiments agreed very well to the ones obtained from the elastic multipole method \( (n_{max} = 10) \) and finite element simulations for all three configurations of holes (see Fig. 10a-f). We also compared the equivalent von Mises strain field defined as \( \varepsilon_{vM} = \frac{\sigma_{vM}}{E} = \frac{1}{1+\nu} \sqrt{\varepsilon_{xx}^2 - \varepsilon_{xx} \varepsilon_{yy} + \varepsilon_{yy}^2 + 3\varepsilon_{xy}^2 + \frac{\nu}{(1-\nu)^2} (\varepsilon_{xx} + \varepsilon_{yy})^2} \) that were obtained from the elastic multipole method \( (n_{max} = 10) \), finite element simulations and from the DIC analyses of experiments (see Fig. 11). For all three configurations of holes the strain fields agree very well between the elastic multipole method (Fig. 11a-c) and finite element simulations (Fig. 11d-f). The strain fields for experimental samples are qualitatively similar, but they deviate quantitatively near the holes as can be seen from heat maps.
FIG. 10. Uniaxial vertical compression of elastic structures with holes and inclusions. (a-c) Schematic images describing the initial undeformed shapes of structures with two holes in three different configurations (horizontal, vertical and inclined at 45°) and the direction of applied external strain $\varepsilon^\text{ext}_{yy} = -0.05$. (d-f) Deformed contours of holes obtained with the elastic multipole method ($n_{\text{max}} = 10$, blue dashed lines), finite element simulations (solid green lines) and experiments (red solid lines). (g-i) Schematic images describing the initial undeformed shapes of structures with one hole (white circles) and one inclusion (orange circles) in three different configurations (horizontal, vertical and inclined at 45°), and the direction of applied external strain $\varepsilon^\text{ext}_{yy} = -0.05$. (j-l) Deformed contours of holes and inclusions obtained with the elastic multipole method ($n_{\text{max}} = 10$, blue dashed lines), finite element simulations (solid green lines) and experiments (red solid lines). In all cases, the size of specimens was 100 mm $\times$ 100 mm $\times$ 25 mm, the diameters of each hole/inclusion were $d = 8.11$ mm, and their separation distances were $a = 9.50$ mm.

(Fig. [11g-i]). The quantitative comparison of strains at four different points A-D (marked in Fig. [11]) showed a relative error of $1-4\%$ between the elastic multipole method and finite elements, and a relative error of $0-15\%$ between the elastic multipole method and experiments (see Table [IV]). The discrepancy between the elastic multipole method and finite element simulations is attributed to the finite size effects. For the elastic multipole method we assumed an infinite domain, while finite domains were modeled in finite element simulations to mimic experiments. Because the domains
FIG. 11. Equivalent von Mises strain fields $\varepsilon_{vM}$ (see Eq. (37)) for uniaxially vertically compressed elastic structures with two holes (white circles) in three different configurations (horizontal, vertical, inclined at 45°) introduced in Fig. 10. Equivalent von Mises strain fields $\varepsilon_{vM}$ were obtained with (a-c) the elastic multipole method ($n_{\text{max}} = 10$), (d-f) finite element simulations, and (g-i) the DIC analysis of experiments. Note that the strain data was corrupted near the edges for some samples due to the imperfections in speckle patterns near the boundary. For this reason, we omitted the affected border regions (grey frames) in heat maps (g-i). Four marked points A-D are chosen for the quantitative comparison of strains $\varepsilon_{vM}$ that are reported in Table IV.

are relatively small, interactions of induced elastic multipoles with boundaries become important, which is discussed in details in the companion paper [43]. The discrepancy between experiments and the elastic multipole method is also attributed to the confounding effects of non-linear deformation due to moderately large compression ($\varepsilon_{\text{ext}} = -0.05$), 3D deformation due to relatively thick samples, and experimental limits coming from the manufacturing imperfections, nonzero friction between the specimen and the mounting grips of the testing machine, the alignment of camera with the specimen (2D DIC system was used), as well as the errors coming from the choice of DIC parameters (see e.g. [53, 54]).

Experiments were repeated with relatively rigid inclusions ($E_{\text{inc}}/E_0 = 3000$), where the specimens described above were reused. Acrylic (PMMA) rods were inserted in one of the holes and glued with a cyanoacrylate glue for each of the specimen. The contours of deformed holes obtained in experiments agreed very well to the ones obtained from the
FIG. 12. Equivalent von Mises strain fields \( \epsilon_{vM} \) (see Eq. (37)) for uniaxially vertically compressed elastic structures with one hole (white circles) and one inclusion (blue circles) in three different configurations (horizontal, vertical, inclined at 45°) introduced in Fig. 10. Equivalent von Mises strain fields \( \epsilon_{vM} \) were obtained with (a-c) the elastic multipole method \((n_{\text{max}} = 10)\), (d-f) finite element simulations, and (g-i) the DIC analysis of experiments. Note that the strain data was corrupted near the edges for some samples due to the imperfections in speckle patterns near the boundary. For this reason, we omitted the affected border regions (grey frames) in heat maps (g-i). Four marked points A-D are chosen for the quantitative comparison of strains \( \epsilon_{vM} \) that are reported in Table V.

The elastic multipole method \((n_{\text{max}} = 10)\) and finite element simulations for all three configurations of holes and inclusions (see Fig. 10j-l). A relatively good agreement was also obtained for strain fields (see Fig. 12), where the strains inside rigid inclusions are very small (dark blue color). The quantitative comparison of strains at four different points A-D (marked in Fig. 12) showed a relative error of 0-5% between the elastic multipole method and finite elements, and a relative error of 0-12% between the elastic multipole method and experiments (see Table V).
TABLE IV. Quantitative comparison for the values of equivalent von Mises strains \( \varepsilon_{vM} \) at points A-D (defined in Fig. 11) in compressed samples with two holes in three different configurations (horizontal, vertical, inclined) obtained with the elastic multipole method (EMP), finite element simulations (FEM) and the DIC analysis of experiments (EXP). The relative percent errors between the EMP and FEM were calculated as 100 \( \times \) \( (\varepsilon_{vM}^{(EMP)} - \varepsilon_{vM}^{(FEM)}) / \varepsilon_{vM}^{(FEM)} \). The relative percent errors between the EMP and EXP were calculated as 100 \( \times \) \( (\varepsilon_{vM}^{(EMP)} - \varepsilon_{vM}^{(EXP)}) / \varepsilon_{vM}^{(EXP)} \). The relative percent errors between EMP and EXP were calculated as 100 \( \times \) \( (\varepsilon_{vM}^{(EXP)} - \varepsilon_{vM}^{(EMP)}) / \varepsilon_{vM}^{(EXP)} \).

<table>
<thead>
<tr>
<th>Points</th>
<th>Strain ( \varepsilon_{vM} ) (%)</th>
<th>Error of EMP (%)</th>
<th>Strain ( \varepsilon_{vM} ) (%)</th>
<th>Error of EMP (%)</th>
<th>Strain ( \varepsilon_{vM} ) (%)</th>
<th>Error of EMP (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>EMP 5.85</td>
<td>FEM 5.70</td>
<td>EXP 5.84</td>
<td>2.63</td>
<td>0.17</td>
<td>EMP 5.72</td>
</tr>
<tr>
<td>B</td>
<td>EMP 3.55</td>
<td>FEM 3.21</td>
<td>EXP 3.56</td>
<td>2.69</td>
<td>3.78</td>
<td>EMP 3.54</td>
</tr>
<tr>
<td>C</td>
<td>EMP 5.27</td>
<td>FEM 5.14</td>
<td>EXP 4.62</td>
<td>2.53</td>
<td>14.07</td>
<td>EMP 6.16</td>
</tr>
<tr>
<td>D</td>
<td>EMP 5.14</td>
<td>FEM 5.00</td>
<td>EXP 4.92</td>
<td>2.80</td>
<td>4.47</td>
<td>EMP 5.10</td>
</tr>
</tbody>
</table>

TABLE V. Quantitative comparison for the values of equivalent von Mises strains \( \varepsilon_{vM} \) at points A-D (defined in Fig. 12) in compressed samples with one hole and one inclusion in three different configurations (horizontal, vertical, inclined) obtained with the elastic multipole method (EMP), finite element simulations (FEM) and the DIC analysis of experiments (EXP). The relative percent errors between the EMP and FEM were calculated as 100 \( \times \) \( (\varepsilon_{vM}^{(EMP)} - \varepsilon_{vM}^{(FEM)}) / \varepsilon_{vM}^{(FEM)} \). The relative percent errors between the EMP and EXP were calculated as 100 \( \times \) \( (\varepsilon_{vM}^{(EXP)} - \varepsilon_{vM}^{(EMP)}) / \varepsilon_{vM}^{(EXP)} \). The relative percent errors between EMP and EXP were calculated as 100 \( \times \) \( (\varepsilon_{vM}^{(EXP)} - \varepsilon_{vM}^{(EMP)}) / \varepsilon_{vM}^{(EXP)} \).

<table>
<thead>
<tr>
<th>Points</th>
<th>Strain ( \varepsilon_{vM} ) (%)</th>
<th>Error of EMP (%)</th>
<th>Strain ( \varepsilon_{vM} ) (%)</th>
<th>Error of EMP (%)</th>
<th>Strain ( \varepsilon_{vM} ) (%)</th>
<th>Error of EMP (%)</th>
</tr>
</thead>
<tbody>
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<td>A</td>
<td>EMP 6.06</td>
<td>FEM 6.02</td>
<td>EXP 6.13</td>
<td>0.66</td>
<td>1.14</td>
<td>EMP 5.54</td>
</tr>
<tr>
<td>B</td>
<td>EMP 3.48</td>
<td>FEM 3.50</td>
<td>EXP 3.11</td>
<td>0.57</td>
<td>11.90</td>
<td>EMP 5.72</td>
</tr>
<tr>
<td>C</td>
<td>EMP 5.68</td>
<td>FEM 5.64</td>
<td>EXP 5.46</td>
<td>0.71</td>
<td>4.03</td>
<td>EMP 2.84</td>
</tr>
<tr>
<td>D</td>
<td>EMP 5.03</td>
<td>FEM 5.02</td>
<td>EXP 4.85</td>
<td>0.20</td>
<td>3.71</td>
<td>EMP 5.02</td>
</tr>
</tbody>
</table>

IV. CONCLUSION

In this paper we demonstrated how the induction and multipole expansion, which are common concepts in electrostatics, can be effectively used also for analyzing the linear deformation of infinite 2D elastic structures with circular holes and inclusions for both plane stress and plane strain conditions. Unlike in electrostatics, where dipoles are the lowest order of induced multipoles, in elasticity quadrupoles are the lowest order of induced multipoles due to external load. Dipoles in elasticity are topological defects called dislocations [45] and thus they cannot be induced by external loads. Note also that in contrast to electrostatics there are two types of quadrupoles \( (Q, P) \) in elasticity due to the biharmonic nature of the Airy stress function.

The multipole expansion is a so called far-field method and hence it is extremely efficient when holes and inclusions are far apart. In this case very accurate results can be obtained by considering only induced quadrupoles, because the effect of higher order multipoles decays more rapidly at large distances. When holes and inclusions are closer together, their interactions via higher order multipoles become important as well. The accuracy of the results increases exponentially with the maximum degree of elastic multipoles, which is also the case in electrostatics and is characteristic for spectral methods [50].

The elastic multipole method presented here was restricted to deformations of infinite structures with holes and inclusions of circular shapes. It can be generalized to deformations of finite size structures by employing the concept of image charges from electrostatics, which is discussed in detail in the companion paper [43]. This method can also be adapted to describe deformations of structures with non-circular holes and inclusions, and can in principle be generalized to describe deformations of curved thin shells with inclusions.

While the elastic multipole method presented here focused only on the linear deformation, similar concepts can also be useful for describing the postbuckling deformation of mechanical metamaterials. Previously it was demonstrated that the buckled patterns of structures with periodic arrays of holes [15, 19, 55, 56], square frames [57, 58], and kirigami slits [57] can be qualitatively described with interacting quadrupoles. Furthermore, the approach with elastic multipoles was recently extended to enable the full nonlinear description of compressed structures with periodic arrays of holes, which can predict the initial linear deformation, the critical buckling load, as well as the post-buckling deformation [59]. Thus the elastic multipole method has the potential to significantly advance our understanding of deformation patterns in structures with holes and inclusions.
ACKNOWLEDGEMENTS

This work has been supported by NSF through the Career Award DMR-1752100 and by Slovenian Research Agency through the grant P2-0263. We would like to acknowledge useful discussions with Michael Moshe (Hebrew University) and thank Jonas Trojer (University of Ljubljana) for help with experiments.

Appendix A: Finite element simulations

Linear analyses in finite element simulations were performed with the commercial software Ansys® Mechanical, Release 17.2. Geometric models of plates with holes and inclusions were discretized with 2D eight-node, quadratic elements of type PLANE183 set to the plane stress state option. The material for plates and inclusions was modeled as a linear isotropic elastic material. In order to minimize the effect of boundaries for the comparison with the elastic multipole method, which considers an infinite domain, we chose a sufficiently large square shaped domain of size \( L = 400d \), where \( d \) is the diameter of inclusions. To ensure high accuracy, we used a fine mesh with 360 quadratic elements evenly spaced around the circumference of each inclusion. To keep the total number of elements at a manageable number, the size of elements increased at a rate of 2% per element, when moving away from inclusions, up to the largest elements on the domain boundaries with the edge length of \( L/200 \). Overall the mesh comprised of approximately 250,000 elements with \( \approx 750,000 \) nodes having a total of \( \approx 1,500,000 \) degrees of freedom (each node has 2 degrees of freedom as it can move in the \( x \) and \( y \) directions). To prevent rigid body motions of the whole structure, we fixed the following 3 degrees of freedom: displacement vector at the center of the square domain were specified to be zero \((u_x(0,0) = 0, u_y(0,0) = 0)\); the midpoint of the left edge of the square domain was constrained to move only in the \( x \) direction \((u_y(-L/2,0) = 0)\). For consistency with finite element simulations we imposed the same set of constraints \((u_x(0,0) = 0, u_y(0,0) = 0, u_y(-L/2,0) = 0)\) for the elastic multipole method. This was done in two steps. After obtaining the displacement field \((u_{x,\text{mult}}^\text{mult}(x,y), u_{y,\text{mult}}^\text{mult}(x,y))\) from the elastic multipole method, we first subtracted the displacement \((u_{x,\text{mult}}^\text{mult}(0,0), u_{y,\text{mult}}^\text{mult}(0,0))\) at every point

\[
\begin{align*}
    u_{x,\text{mult}}^\text{mult}(x,y) &= u_{x,\text{mult}}^\text{mult}(x,y) - u_{x,\text{mult}}^\text{mult}(0,0), \\
    u_{y,\text{mult}}^\text{mult}(x,y) &= u_{y,\text{mult}}^\text{mult}(x,y) - u_{y,\text{mult}}^\text{mult}(0,0),
\end{align*}
\]

(A1a)

(A1b)

to ensure that the center of the square domain is fixed \((u_{x,\text{mult}}^\text{mult}(0,0) = u_{y,\text{mult}}^\text{mult}(0,0) = 0)\). For this updated displacement field, the coordinates of points in the deformed configuration are \(x'(x,y) = x + u_{x,\text{mult}}^\text{mult}(x,y)\) and \(y'(x,y) = y + u_{y,\text{mult}}^\text{mult}(x,y)\). To impose the last constraint \((u_y(-L/2,0) = 0)\), this new deformed configuration was then rotated anticlockwise by the angle \(\theta = \tan^{-1}(u_{y,\text{mult}}^\text{mult}(-L/2,0)/[L/2 - u_{x,\text{mult}}^\text{mult}(-L/2,0)])\) around the center of coordinate system as

\[
\begin{align*}
    x''(x,y) &= x'(x,y) \cos \theta - y'(x,y) \sin \theta \equiv x + u_{x,\text{mult}}^\text{mult}(x,y), \\
    y''(x,y) &= x'(x,y) \sin \theta + y'(x,y) \cos \theta \equiv y + u_{y,\text{mult}}^\text{mult}(x,y).
\end{align*}
\]

(A2a)

(A2b)

The set of displacement fields \(u_{x,\text{mult}}^\text{mult}(x,y)\) and \(u_{y,\text{mult}}^\text{mult}(x,y)\) was then used for comparison with finite element simulations.


[59] Private communication with M. Moshe.