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On the existence of equilibria to Bayesian games with non-finite type and action spaces

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Abstract

Equilibria are shown to exist in non-finite Bayesian games if the type and action spaces are compact and convex subsets of finite dimensional Euclidean space, utility functions are continuous, expected utility functions are strictly quasiconcave in the agent's action, the set of rationalizable mappings have a uniformly bounded slope and posterior beliefs are suitably continuous.

Keywords: Bayesian Nash equilibria; Bayesian games; Equilibrium existence

JEL classification: C62; C72

General conditions for the existence of pure strategy Nash equilibria to games in which the strategy spaces are finite dimensional are well known (Dasgupta and Maskin, 1986; Debreu, 1952; Glicksberg, 1952; Fan, 1952). A Bayesian game in which the type spaces are finite can be redefined as a normal form game in which the strategy spaces are finite dimensional. Alternatively, in some applications in which the type spaces are non-finite subsets of Euclidean space and the action spaces are finite, monotonicity conditions on the preferences imply that the best-responses can be thought of as living in a finite dimensional space. However, if the type and action spaces are both non-finite then the best responses are elements of an infinite dimensional space. While Brouwer's and Kakutani's fixed point theorems apply to arbitrary topological spaces, establishing that an application satisfies conditions for which the best response mapping satisfies the topological conditions of these theorems may be quite difficult. Specifically, compactness of sets (as required by the Brouwer and Kakutani fixed point

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theorems) is a difficult concept to work with in infinite dimensional spaces.[1] Our solution is to rely on compactness of the finite dimensional action spaces and invoke Shauder's fixed point theorem.

This approach is not novel in economics. In several applications pure strategy equilibria to Bayesian games with non-finite type and action spaces have been shown to exist. Fey (2001) analyzes an imperfect information two-player rent seeking model and establishes equilibrium existence for certain parameterizations. Meirowitz (2001) analyzes a model of elections in which candidates are uncertain of the other candidate's preferences and establishes existence of equilibria when candidates are risk averse. Moreover, several authors have established existence of equilibria in infinite dimensional general equilibrium settings (e.g. Lucas and Stokey, 1987). However, the useful result appears new.

We show that if type and action spaces are both non-empty, compact and convex subsets of a finite dimensional Euclidean space, agent utility functions are continuous in their type and action as well as the action of the other players, agent expected utility functions are strictly quasiconcave in the agent's action for every type, the set of rationalizable mappings from type to action have uniformly bounded slope and agent posterior beliefs are suitably continuous in their types then a pure strategy Bayesian Nash equilibrium exists.[2] While equilibrium existence hinges on finding a fixed point of an operator on an infinite dimensional space, verification that an application satisfies the sufficient conditions of our result only requires inspection of finite dimensional sets and functions on finite dimensional sets.

We consider a Bayesian game involving a finite set of agents N. Each agent has a type $\theta_i \in \Theta_i \subset \mathbb{R}^{d_i}$ and an action space $A_i \subset \mathbb{R}^{z_i}$. A profile of types is a vector $\theta = (\theta_1, \ldots, \theta_n) \in \Theta := x_{i \in N} \Theta_i$ and a profile of actions is a vector $a = (a_1, \ldots, a_n) \in A := x_{i \in N} A_i$. We denote the vector of types or actions that exclude agent i by θ_{-i} and a_{-i}. We use the notation $\| \cdot \|_R$ to denote the Euclidean norm. Conditional on her type θ_i agent i's posterior belief about θ_{-i} is represented by the probability measure $\eta_i(\cdot | \theta_i)$.[3] Agent preferences are represented by utility functions $u_i(a; \theta): A \times \Theta \to \mathbb{R}^1$. A strategy for an agent is then a measurable function $s_i(\theta_i): \Theta_i \to A_i$. By F_i we denote the space of measurable functions from Θ_i into A_i endowed with the max-norm. Thus if $f_i \in F_i$ then

$$\|f_i\|_m = \max_{\theta_i \in \Theta_i} \|f(\theta_i)\|_R.$$

By C_i we denote the space of continuous such functions endowed with the same norm. By F and C we denote the product spaces $\prod_{i \in N} F_i$ and $\prod_{i \in N} C_i$, respectively. We also refer to the product spaces $F_{-i} = \prod_{j \in N \setminus i} F_j$ and $C_{-i} = \prod_{j \in N \setminus i} C_j$. We use the notation $\| \cdot \|_p$ to denote the product norm in the relevant product spaces.[4] The standard equilibrium concept for this class of games is Bayesian Nash equilibria.

[1] Recall that a closed unit ball is compact in a Banach space only if the space is finite dimensional.

[2] By rationalizable, we mean optimal for some profile of mappings for the other players.

[3] We suppress the prior beliefs about θ and just refer to the conditional posteriors as this is the necessary measure to express the objective functions for agents. Accordingly, the only limit we have imposed on the underlying probability measure on the state variable is that the conditional probability measure exists. In the proposition we state a continuity requirement for this probability measure.

[4] Thus given $(f_1, f_2) \in F = F_1 \times F_2$ the norm is $\|(f_1, f_2)\|_p = \max\{\|f_1\|_m, \|f_2\|_m\}$. Many other norms induce the same topology in the product space.

Definition 1. A pure strategy Bayesian Nash equilibrium is an element $f=(f_1,\ldots,f_n)\in F$ s.t. for every $i\in N$

$$f_i(\theta_i)\in\arg\max_{a_i\in A_i}\left\{\int u_i(a_i, f_{-i}(\theta_{-i}); \theta_i, \theta_{-i})\,\mathrm{d}\eta_i(\theta_{-i}|\theta_i)\right\}. \quad (1)$$

The result which we shall prove is

Proposition 1. *A Bayesian game has a pure strategy BNE if for each $i\in N$*

1. *Θ_i and A_i are nonempty, convex and compact subsets of Euclidean space.*
2. *$u_i(a,\theta)$ is continuous.*
3. *For every θ and measurable function $f_{-i}(\theta_{-i})$*

$$\int u_i(a_i, f_{-i}(\theta_{-i}); \theta_i, \theta_{-i})\,\mathrm{d}\eta_i(\theta_{-i}|\theta_i)$$

is strictly quasiconcave in a_i.

4. *For every $\varepsilon_i>0$ there exists some constant δ_i s.t. if*

$$g_i(\theta_i)\in\arg\max_{a_i\in A_i}\left\{\int u_i(a_i, f_{-i}(\theta_{-i}); \theta_i, \theta_{-i})\,\mathrm{d}\eta_i(\theta_{-i}|\theta_i)\right\}$$

for some $f_{-i}(\theta_{-i})\in F_{-i}$ then $\sup_{\{(\theta_i', \theta_i'')\in\Theta_i:\|\theta_i'-\theta_i''\|<\delta_i\}}\|g(\theta_i')-g(\theta_i'')\|_R<\varepsilon_i$.

5. *For a.e. measurable set $A\subset\Theta_{-i}$, $\eta_i(A|\theta_i)$ is continuous in θ_i.*

Conditions 1 and 2 are standard. Condition 3 is satisfied, for example, if for every θ and measurable function $f_{-i}(\theta_{-i})$ the utility function $u_i(a_i, f_{-i}(\theta_{-i}); \theta_i, \theta_{-i})$ is strictly concave in a_i. Condition 4 requires that for each player the set of strategies that are rationalizable (as best responses to some profile of pure strategies f_{-i}) have uniformly bounded slope. Condition 4 is the least natural or technical of these assumptions. In applications it is this condition which is likely to fail or be difficult to verify. Condition 5 requires that an agent's belief about the types of the other players is continuous in the agent's private information/type. When agent types are independent the condition is trivially satisfied. Alternatively, if there is an underlying state variable with a common prior distribution and the individual types are generated by a known state conditional distribution, condition 4 requires that Bayes' rule yield a posterior distribution that is continuous in the agent's type.

The proof of the proposition involves establishing the existence of a fixed-point to a mapping from a functional space into itself. Since compactness of sets is a difficult concept to work with when dealing with function spaces and compactness of operators mapping a function space into itself is not difficult to establish, Shauder's fixed point theorem (of which Brouwer's fixed point theorem is a special case) is the useful result (Zeideler, 1985).

Theorem 1. ((Shauder's fixed point theorem, 1930)) *If M is a nonempty, closed, bounded, convex subset of a Banach space and T: $M\to M$ is a compact operator then T has a fixed point.*

We begin by defining the relevant correspondence. By $\xi_i(f_{-i})$: $F_{-i}\to\to F_i$ we denote the best response correspondence for agent i. This correspondence is defined by the problem in Eq. (1). When we wish to denote the set of optimal actions a_i for a given pair (θ_i, f_{-i}) we write $\xi_i(\theta, f_{-i})$. This is a subset of A_i. When we wish to denote the correspondence that maps θ_i into optimal actions holding fixed f_{-i} we write $\xi_i(f_{-i})$. By $\xi(f)$: $F \rightarrow \rightarrow F$ defined by

$$\xi(f) = (\xi_1(f_{-1}), \ldots, \xi_n(f_{-n})) \tag{2}$$

we denote the best response correspondence. A pure strategy Bayesian Nash equilibrium is a fixed point to this mapping. Before proving the proposition we present a definition and a second result which will be of use.

Definition 2. An operator is compact if it is continuous and it maps bounded sets into relatively compact sets.

The second result offers a very useful characterization of relative compactness in C. We state the theorem in the notation of the current paper.

Theorem 2. (Arzela–Ascoli theorem) *A set $D \subset C$ is relatively compact iff two conditions hold*: *uniform boundedness,*

$$\sup_{f \in D} (\sup_{\theta \in \Theta} \|f(\theta)\|_p) < \infty$$

and equicontinuity

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \text{ s.t. } \sup_{f \in D} \|f(\theta') - f(\theta'')\|_p < \varepsilon \text{ whenever } \|\theta' - \theta''\|_R < \delta.$$

We can now prove the proposition.

Proof. Overview of the proof: assume that the five conditions in the proposition are satisfied. The proof proceeds in four steps. In Step 1 we show that for every $i \in N$, and every $f_i \in F_i$, $\xi_i(\theta_i, f_{-i})$ is a singleton and that this choice varies continuously in θ_i, and thus the solution to (1) is a continuous function $\xi_i(\theta_i, f_{-i})$: $\Theta_i \rightarrow A_i$. This means that $\xi_i(f_{-i})$: $F_{-i} \rightarrow \rightarrow F_i$ is actually of the form $\xi_i(f_{-i})$: $F_{-i} \rightarrow C_i$. This implies that any fixed-point to the operator

$$\xi(f): C \rightarrow C \tag{3}$$

is a Bayesian Nash equilibrium. In Step 2 we show that the operator in (3) is continuous.

In Step 3 we show that the fact that the operator in (3) maps from a space of uniformly continuous and bounded functions into itself means that it satisfies uniform boundedness and condition 4 of the theorem implies that equicontinuity is satisfied. In Step 4 we combine the result from Step 3 and Theorem 2 to attain the fact that the operator in (3) is a compact operator. Since C is a nonempty, closed, bounded, convex subset of a Banach space, the existence of a fixed-point to $\xi(f)$: $C \rightarrow C$ follows from Step 2 and Theorem 1.

Step 1. Conditions 1 and 2 of the proposition imply that for every $i \in N$ and every $\theta_i \in \Theta_i$ and every $f_{-i} \in F_{-i}$ solving (1) involves optimization of a continuous function over a nonempty, convex and compact set. Moreover conditions 2 and 5 state that the objective function is continuous in the parameter θ_i. The theorem of the maximum (Berge, 1963) implies that the optimal action a_i exists for every $\theta_i \in \Theta_i$ and every $f_{-i} \in F_i$ and that the correspondence $\xi_i(\theta_i, f_{-i})$: $\Theta_i \rightarrow \rightarrow A_i$ is upper hemicontinuous. Condition 3 requires that the objective function is strictly quasiconcave in the choice variable and thus since A_i is convex, the optimal action is unique for each $\theta_i \in \Theta_i$ and $f_{-i} \in F_i$. This means that for each $i \in N$ and every $f_{-i} \in F_{-i}$ the best response for agent i is a continuous function of θ_i. Thus for every $f_{-i} \in F_{-i}$, $\xi_i(f_{-i}) \in C_i$. We thus focus only on $f \in C$.

Step 2. By condition 2 of the proposition for any $i \in N$ and any $\theta_i \in \Theta_i$ the objective function in (1) is a continuous function of f_{-i} on C_{-i}. We refer to continuity relative to the product topology on the product space C. This and condition 1 of the proposition implies that for any $i \in N$ and $\theta_i \in \Theta_i$ the objective function in (1) is a continuous function of f_{-i} on C_{-i}, and the choice variable a_i is chosen from a compact set. Thus the theorem of the maximum implies that for any θ_i as a function of f_{-i} the optimal action is upper hemicontinuous. Thus the correspondence $\xi_i(\theta_i, f_{-i})$: $C_{-i} \rightarrow \rightarrow A_i$ is upper hemicontinuous. Since we have shown that the optimal action is unique and continuous in θ_i, this implies that we have the continuous function $\xi_i(\theta_i, f_{-i})$: $\Theta_i \times C_{-i} \rightarrow A_i$. This means that $\xi_i(f_{-i})$: $C_{-i} \rightarrow C_i$ is continuous for each $i \in N$ and thus $\xi(f)$: $C \rightarrow C$ is a continuous operator.

Step 3. In order to invoke Theorem 2 and establish that the operator in (3) is a compact operator, we show that (i) C satisfies boundedness and (ii) for any $D \subset C$ the image of D through ξ, $\xi(D) \subset C$ satisfies equicontinuity. Since A_i is compact (condition 1 of the proposition) the function $\xi_i(\theta_i, f_{-i})$: $\Theta_i \rightarrow A$ is bounded and thus C_i satisfies uniform boundedness for each $i \in N$, and thus C satisfies uniform boundedness—thus (i) is established. Condition 4 requires that for each i $\forall \varepsilon_i > 0$ $\exists \delta_i(\varepsilon_i) > 0$ s.t. $\sup_{f_i \in \xi_i(D)} \| f_i(\theta_i') - f_i(\theta_i'') \|_p < \varepsilon_i$ whenever $\|\theta_i' - \theta_i''\|_R < \delta_i(\varepsilon_i)$. For any ε letting $\delta = \min_i \delta_i(\varepsilon)$ the above implies that $\xi(D)$ is equicontinuous—thus (ii) is established.

Step 4. Since for each $i \in N$, C_i is a closed, bounded and convex Banach space so is the product space C. By Definition 2 and Step 3 the operator $\xi(f)$: $C \rightarrow C$ is compact. By Step 2 this operator is continuous. Thus, by Theorem 1 a fixed point to (3) exists. □

The requirement that expected utility be strictly quasiconcave allows us to conclude that we have the continuous functions $\xi_i(\theta_i, f_{-i})$: $\Theta_i \times C_{-i} \rightarrow A_i$ in Step 2. This ensures that we are finding the fixed point to a single-valued mapping (operator). With only quasiconcavity this mapping would be an upper hemicontinuous correspondence. In some cases it may be possible to make a selection which yields a continuous operator. The requirement that the expected utility is quasiconcave may be replaced with stronger assumptions on the primitive state contingent utility function $u_i(a_i, a_{-i}, \theta)$. For example if this function is strictly concave in a_i then the expected utility function will be strictly concave and the proof will go through. The assumption that there is a uniform bound on the slope of rationalizable strategies is used in establishing that the set of best responses is equicontinuous. This condition is of course not necessary. For example if the set of strategies that are best responses to strategy profiles that involve rationalizable strategies have a uniformly bounded slope then the result will attain. While the current result may not isolate the weakest sufficient conditions for equilibrium existence in Bayesian games with non-finite type and action spaces, the five sufficient conditions may often be easy to verify in applications. Moreover, the proof is suggestive of how specific arguments can be constructed when the conditions of Proposition 1 are not satisfied.

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