Soc Choice Welfare (2004) 22: 1–12
DOI: 10.1007/s00355-003-0227-9

Social Choice and Welfare
© Springer-Verlag 2004

The core of social choice problems with monotone preferences and a feasibility constraint

Adam Meirowitz

Princeton University, Department of Politics, Corwin Hall, Princeton, NJ 08544, USA
(e-mail: ameirowi@princeton.edu)

Received: 11 February 2002 / Accepted: 5 February 2003

Abstract. We consider collective choice with agents possessing strictly monotone, strictly convex and continuous preferences over a compact and convex constraint set contained in $\mathbb{R}^k_+$. If it is non-empty the core will lie on the efficient boundary of the constraint set and any policy not in the core is beaten by some policy on the efficient boundary. It is possible to translate the collective choice problem on this efficient boundary to another social choice problem on a compact and convex subset of $\mathbb{R}^c_+$ $(c < k)$ with strictly convex and continuous preferences. In this setting the dimensionality results in Banks (1995) and Saari (1997) apply to the dimensionality of the boundary of the constraint set (which is lower than the dimensionality of the choice space by at least one). If the constraint set is not convex then the translated lower dimensional problem does not necessarily involve strict convexity of preferences but the dimensionality of the problem is still lower. Broadly, the results show that the homogeneity afforded by strict monotonicity of preferences and a compact constraint set makes generic core non existence slightly less common. One example of the results is that if preferences are strictly monotone and convex on $\mathbb{R}^2$ then choice on a compact and convex constraint exhibits a version of the median voter theorem.

1 Introduction

With overwhelming regularity scholars of public opinion report that voters "want to have their cake and eat it too" (Zaller 1998). Voters want lower taxes and more social spending without increasing the deficit; travelers want increased airline safety without restrictions on civil liberties. Accordingly,

I thank Donald Saari for helpful comments on an earlier version of this paper.

3 5 5 0 0 2 2 7 B
Journal number | Manuscript number
Dispatch: 4.2.2004 Journal: Social Choice and Welfare No. of pages: 12
Author's disk received ☑ Used ☑ Corrupted ☐ Mismatch ☐ Keyed ☐

in some choice contexts public sentiment seems to exhibit monotonicity–the preference for more of each issue dimension. Preference heterogeneity surfaces in the form of differing marginal rates of substitution. Despite preferences for more, policy choice often involves trade-offs. A relevant binding constraint on feasible policies exists. While these observations are not inconsistent with many theories of preference aggregation, they do serve to restrict the space of preferences and choice spaces in an interesting manner. Specifically, monotonicity can be viewed as a limit on the degree of preference heterogeneity.

Standard approaches to social choice theory consist of uncovering the import of restrictions on (1) the set of individual orderings and (2) the size or structure of the policy space. In the finite dimension setting one approach has been to isolate bounds on the dimensionality of the policy space which ensure that stable core points can exist. Plott's symmetry conditions demonstrate that a policy is stable if it is some agent's ideal policy and it is possible to pair off agents with opposing gradient vectors (Plott 1967). However, this construction does not generally yield robust or stable examples. McKelvey and Schofield (1986, 1987) identified critical dimensions for generic core non-existence. Banks (1995) corrected a mistake in this work and Saari (1997) found sharp bounds closing the question. In this paper we examine the extent to which core emptiness results are effected by the order restriction of monotonicity and the outcome restriction of a constrained choice set lying in a finite dimensional space. The main results demonstrate an equivalence between a social choice problem with monotone preferences on a compact choice set and a lower dimensional social choice problem without monotone preferences. Accordingly, the restrictions of monotonicity and constrained choice make it slightly easier to generate stable core points.

To motivate the analysis, we start with the example of a household of three choosing a consumption bundle under a voting rule. Let agents $N = \{1, 2, 3\}$ each have Cobb-Douglas utility over $\mathbb{R}^2_+$. Agent i's utility function is then $u_i(x^1, x^2) = \alpha_i \ln(x^1) + (1 - \alpha_i) \ln(x^2)$ with $0 < \alpha_1 < \alpha_2 < \alpha_3 < 1$. In this case no point is optimal for each agent, and no point is stable (or unbeaten) under majority rule. The majority rule core is empty. In this 2-dimensional problem even the unanimity rule core is empty.[1] Suppose now that our family faces a non linear budget constraint of the form $B = \{x \in \mathbb{R}^2_+ : \|x\| < 1\}$. Here each agents constrained optimum is not well defined, and no point in B is stable under unanimity rule. However, if we have $B = \{x \in \mathbb{R}^2_+ : \|x\| \leq 1\}$ then the problem has very different core properties. Since B is compact now each agent's constrained optimum exists. Moreover, under unanimity rule any point that is not on the efficient boundary $B^+ = \{x \in \mathbb{R}^2_+ : \|x\| = 1\}$ will be beaten by a scaler multiple of itself (with the scaler greater than unity) as

[1] In contrast if preferences are Euclidean then the majority rule core is generically empty and the unanimity rule core corresponds to the Pareto set (triangle with simplices corresponding to the agent ideal points).

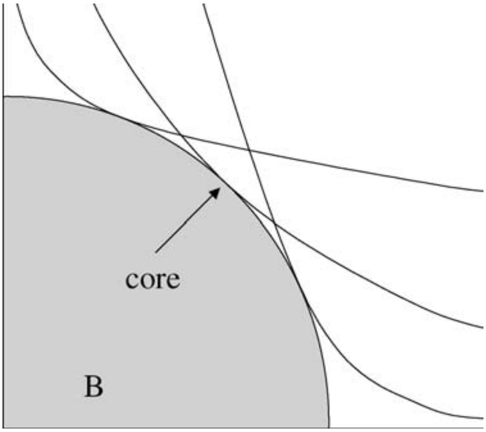

Fig. 1.

more is unanimously preferred to less here. Letting $x_i^* = \arg\max_{x\in B} u_i(x)$, a consequence of our results is: the unanimity rule core will correspond to all the points on the efficient boundary of B that are between x_1^* and x_3^*. Moreover, under majority rule the core is just the point x_2^*. In a setting with monotone preferences introduction of a compact constraint results in a problem that is equivalent to choosing form the boundary of the constraint (which is lower dimensional than the original space). Moreover, even though the boundary of the constraint need not be affine (or convex) if the constraint set itself is convex we can translate the problem of collective choice on the boundary into a well behaved problem of collective choice on a nice convex set.

In this paper we consider social choice problems over $\mathbb{R}_+^k$ involving agents with strictly convex, strictly monotone and continuous preferences. The set of feasible policies is restricted to a compact set. This problem turns out to be equivalent to the social choice problem in which the policy space is a compact subset of a lower dimension Euclidean space with continuous preferences.[2] Moreover, if the constraint is convex the equivalent lower dimensional problem involves strictly convex preferences and a convex choice set. When the dimensionality of the constraint set is 2 the majority rule core is non empty and for an odd number of agents it corresponds to the median of the induced ideal points. Under other simple rules the core is characterized by a set of centrist locations. When the constraint set has dimensionality higher than 2 and preferences are smooth the dimensionality bounds for generic core emptiness from Banks (1995) and Saari (1997) can be relaxed by the codimension of the constraint set as the social choice problem in $\mathbb{R}_+^k$ is transformed into a social choice problem in a lower dimensional space satisfying the conditions in Banks and Saari.

[2] Note, that in this lower dimensional representation of the choice problem preferences are no longer monotone.

2 Environments

We consider collective choice problems involving the set N of n agents and the choice space $\mathbb{R}_+^k$. We denote a point in the space as a vector $x = (x^1, \ldots, x^k)$. Let R_i be a weak ordering on $\mathbb{R}_+^k$, and P_i be the asymmetric ordering induced by R_i. For a particular agent $i \in N$ we define the sets $R_i(x) = \{y \in X : yR_ix\}$; $P_i(x) = \{y \in X : yP_ix\}$; $P^{-1}(x) = \{y \in X : xP_iy\}$. We define two useful orderings on $\mathbb{R}_+^k$: $x \succeq y$ means that $x^d \geq y^d$ for each coordinate d and $x^j > y^j$ for some coordinate j; $x \succ y$ means that $x^d > y^d$ for each coordinate d. We focus on strictly convex, strictly monotone and continuous preferences.

Definition 1. *By $\mathscr{M}$ we denote the set of weak orderings on $\mathbb{R}_+^k$ that satisfy the conditions:*

1. *Strict convexity: for any $x, y \in \mathbb{R}_+^k$ if xRy then for any $\lambda \in (0, 1)$ $\lambda x + (1 - \lambda)yPy$.*
2. *Strict monotonicity: for any $x, y \in \mathbb{R}_+^k$* if *$x \succeq y$ then xPy.*
3. *Continuity: for each $x \in \mathbb{R}_+^k$ the sets $P(x)$ and $P^{-1}(x)$ are open.*

By $\mathscr{M}^n$ we denote the set of n-profiles of strictly convex, strictly monotone and continuous weak orderings and let $\rho \in \mathscr{M}^n$ denote a generic preference profile. We consider constrained choice and assume that the constraint or feasible set $B \subset \mathbb{R}_+^k$ is compact with a non-empty interior. By $\dim(B)$ we denote the dimensionality of the set B.[3] The codimension of B is just $k - \dim(B)$. A social choice problem is then defined by the data; $\langle N, B, \rho, f\rangle$ where f: $\mathscr{M}^n \to \mathscr{B}$ is a simple rule with $\mathscr{B}$ the set of complete and reflexive binary orderings on $\mathbb{R}_+^k$. Given a preference aggregation rule f and preference profile ρ the social ordering is denoted $R_{f(\rho)}$. A preference aggregation rule is called a simple rule if it is defined by its collection of decisive sets. Formally, for a simple rule f there is a proper and monotonic collections of subsets $\mathscr{L}(f) \subset 2^N$ for which $xP_{f(\rho)}y$ iff there exists some $L \in \mathscr{L}(f)$ with xP_iy for all $i \in L$. Such rules are neutral, decisive and monotonic.[4] We will additionally assume that the simple rule is weakly Paretian, meaning that if all agents agree in their preference between two choices then the rule will agree over the two choices (i.e., $N \in \mathscr{L}(f)$). A q-rule is a simple rule in which $\mathscr{L}(f)$ consists of all the coalitions containing at least q members of N. Given a simple rule f, the core is then the set of points in B which are not beaten by any other point in B,

$$C(f, \rho, B) = \{x \in B : P_{f(\rho)}(x) \cap B = \emptyset\}. \tag{1}$$

[3] More precisely B is a $\dim(B)-$dimensional manifold.

[4] See Chapt. 3 of Austen-Smith and Banks (1999) for a discussion of simple rules.

3 Results

3.1 Restricting the problem

Given any $B \subset \mathbb{R}^k_+$ it is not difficult to see that any point in the interior of B is Pareto inefficient. More formally we define the outer edge as $B^+ = \{x \in B : y \succeq x \text{ implies } y \notin B\}$. We use the notation ∂B to denote the boundary of B. We have $B^+ \subset \partial B$. Since for any policy $x \in B \backslash B^+$ there is a policy $y \in B^+$ for which $y \in P_i(x)$ for all $i \in N$ we have the following result.

Lemma 1. *If the simple rule* f *is weakly Paretian then* $C(f, \rho, B) \subset B^+$ *or* $C(f, \rho, B) = \emptyset$.

Proof. By way of a contradiction assume that $C(f, \rho, B) \neq \emptyset$ and there exists a point $x \in C(f, \rho, B) \cap B \backslash B^+$. Since $x \notin B^+$ there exists a $y \in B^+$ s.t. $y \in P_i(x)$ for every $i \in N$. Since f is weakly Paretian this means that $yP_{f(\rho)}x$ contradicting the assumption that $x \in C(f, \rho, B)$. ■

This result implies that we may focus our search for the core on B^+. Moreover, any point $x \in B$ which is beaten by some other policy $z \in B \backslash B^+$, (i.e. $z \in P_{f(\rho)}(x) \cap B \backslash B^+$) must also be beaten by some point $y \in B^+$. This follows from monotonicity, and the fact that if $z \in B \backslash B^+$ there is a point $y \in B^+$ with $y \succeq z$.

Lemma 2. *If the preference aggregation rule* f *is weakly Paretian and* $x \notin C(f, \rho, B)$ *there is a* $y \in B^+$ *s.t.* $y \in P_{f(\rho)}(x)$.

Proof. Assume that $x \notin C(f, \rho, B)$. Either $x \in B^+$ or it is not. In the latter case monotonicity implies that there exists a $y \in B^+$ which is also in $P_i(x)$ for every $i \in N$ and thus $yP_{f(\rho)}x$ and we are finished. In the former case, the fact that $x \notin C(f, \rho, B)$ implies that there is some $y \in B$ for which $yP_{f(\rho)}x$. If $y \in B^+$ we are finished. If not then there is a $z \in B^+$ with $z \succeq y$. Strict monotonicity implies that zP_iy for all $i \in N$. Since we have $yP_{f(\rho)}x$ it must be the case that for some $L \in \mathscr{L}(f)$ yP_ix for every $i \in L$. Transitivity of individual preferences implies that zP_ix for every $i \in L$ and thus $zP_{f(\rho)}x$. Thus we are finished. ■

Combined, these two lemmas imply that to study the core of the problem $\langle N, B, \rho, f \rangle$ it is sufficient to focus on the problem $\langle N, B^+, \rho_{|B^+}, f \rangle$ where $\rho_{|B^+}$ is the restriction of preferences ρ to B^+.

Lemma 3. *If* f *is a weakly Paretian simple rule and* $\rho \in \mathscr{M}^n$ *then* $C(f, \rho, B) = C(f, \rho_{|B^+}, B^+)$.

A topological feature of B^+ will be of value.

Lemma 4. B^+ *is a compact subset of a manifold with dimension of at most* $\dim(B) - 1$.

Proof.

- It is well known that for $g: \mathbb{R}^k \to \mathbb{R}^1$ continuous and B compact $\arg\max_{x \in B}\{g(x)\}$ is a compact set. Since $\succeq$ is a continuous ordering there exists a continuous function g s.t. $B^+ = \arg\max_{x \in B}\{g(x)\}$. Thus B^+ is compact.
- It is well known that the boundary of convex body with a non-empty interior in $\mathbb{R}^k_+$ is homeomorphic to the unit ball S^{k-1} (Theorem 16.4 of Bredon 1993). Thus, ∂B is homeomorphic to $S^{\dim(B)-1}$. Since $B^+ \subset \partial B$ it cannot have dimension higher than $\dim(B) - 1$. ■

While most work (Banks, McKelvey and Ordeshook, Plott, Saari) explicitly assumes that preferences are strictly convex and the choice set is $\mathbb{R}^k$ (or at least a convex set), Saari notes that his analysis and results can be extended to smooth manifolds. Lemma 3 alone implies that a problem $\langle N, B, \rho, f\rangle$ with $\rho \in \mathscr{M}^n$ and B compact (but not necessarily convex) can be transformed into a problem $\left\langle N, B^+, \rho_{|B^+}, f\right\rangle$ in which B^+ is a $\dim(B) - 1$ dimensional manifold. Accordingly for results that do not hinge on convexity Lemma 3 alone is sufficient to show that monotonicity of preferences and a compact constraint result in a problem that is equivalent to social choice on a manifold of dimension $\dim(B) - 1$.

Saari's generic existence results are statements "that examples exist where the conclusion holds even after an example is slightly modified". (p. 222) However, other results like the median voter theorem are statements about the existence of core points for all preference profiles in a particular class. Results of this form tend to require that the collective choice problem involve a convex choice set and strictly convex preferences. Figure 2 exhibits a simple example demonstrating that if B is not convex the restriction of preferences to B^+ may not be single-peaked even though the preferences are strictly convex on $\mathbb{R}^2_+$. The thick line represents ∂B and two sets of indifference curves are exhibited. Here the constraint set is not convex, and as one moves along ∂B the agents' preferences do not exhibit single-peakedness. This is in contrast to the example of Fig. 1 where the constraint was convex.

In the next section we show that if B is convex and $\rho \in \mathscr{M}^n$ then the problem $\langle N, B, \rho, f\rangle$ can be transformed into a problem of the form $\langle N, A, \rho', f\rangle$ where A is a compact and convex subset of $\mathbb{R}^{\dim(B)-1}$ and ρ' is continuous and strictly convex on A. A problem of this type is the subject of the existing core characterization results of Banks and Saari.

3.2 Preserving convexity

While Lemma 4 gives us some topological information about the set B^+ we are primarily interested in the properties of the restriction of ρ to B^+. While B^+ will not be convex, if B is convex it is possible to translate the problem $\langle N, B^+, \rho_{|B^+}, f\rangle$ into a problem $\langle N, A, \rho', f\rangle$ where $A \subset S^{\dim(B)-1}$ is compact and convex and ρ' is strictly convex and continuous (but not strictly monotone) on A. More precisely, let $h(x): \partial B \to S^{\dim(B)-1}$ be a homeomorphism

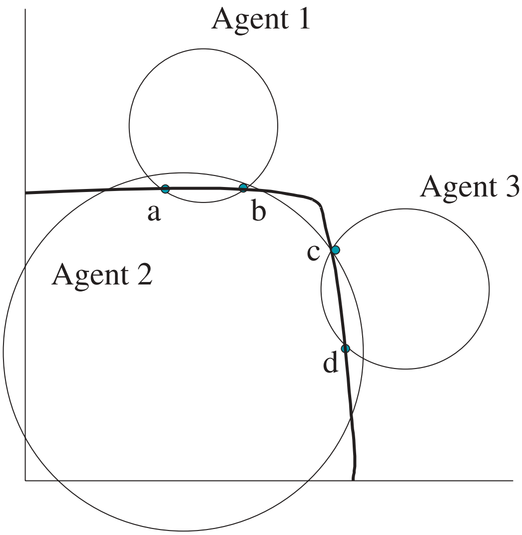

Fig. 2.

and let $h^{-1}(\cdot)$ be the inverse of $h(\cdot)$. Preferences on $S^{\dim(B)-1}$ are defined as follows.

Definition 2. *Given a homeomorphism* $h(x) : \partial B \to S^{\dim(B)-1}$ *and a preference ordering* $R_i \in \mathscr{M}$, *we define the ordering* R_i^h *on* $S^{\dim(B)-1}$ *as:*

$$aR_i^h b \text{ iff } h^{-1}(a)R_i h^{-1}(b).$$

We will show that there is a homeomorphism which maps convex sets into convex sets and that under this mapping R_i^h inherits strict convexity from R_i if B is convex. Before stating and proving the result we exhibit an example that demonstrates the importance of monotonicity. Here monotonicity is violated and the restriction of preferences to ∂B is not strictly convex. Figure 3 illustrates the boundary ∂B with a thick line, and plots the points a, b, c, d on the set B^+. Additionally, indifference curves of three agents are graphed. Agent 1's ideal policy on B^+ lies between a and b. Moreover, agent 1's utility decreases as one moves along the curve B^+ away from this bliss point. A similar statement holds for agent 3 having an ideal point between c and d. However, agent 2 has upper contour sets that are not connected on the curve B^+. Agent 2 prefers a to b and d to c yet she is indifferent between b and c.

When preferences are in $\mathscr{M}$ and B is compact and convex the induced preferences are well behaved.

Proposition 1. *If* B *is compact and convex there exists a compact and convex set* $X \subset S^{\dim(B)-1}$ *and a homeomorphism* $h(x) : B^+ \to X$ *for which the relations* R_i^h *on* X *are strictly convex and continuous for each* $i \in N$.

Proof. Without loss of generality assume that $B \subset S^{\dim(B)}$. Let

$$h(x) = (x^2, x^3, \ldots, x^k). \tag{2}$$

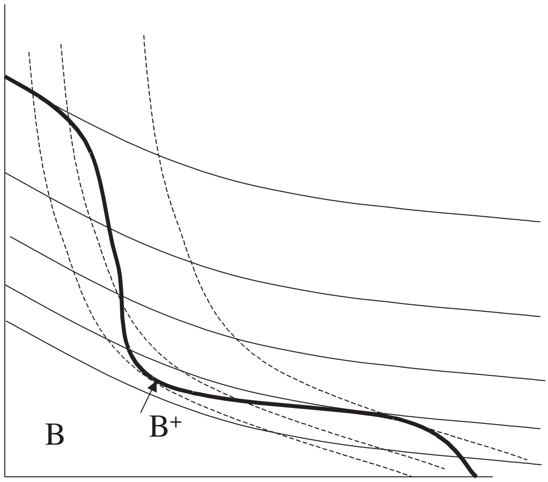

Fig. 3. ---- Agent 1's indifference curves; —— Agent 2's indifference curves

Step 1. This mapping is a projection and is thus continuous. It remains to show that $h^{-1}(\cdot)$ is a continuous function. By definition of B^+ the set

$$h^{-1}((y^2, y^3, \ldots, y^k)) = \{x \in B^+ : (x^2, x^3, \ldots, x^k) = (y^2, y^3, \ldots, y^k)\} \tag{3}$$

is a singleton. If this were not true then the points in $h^{-1}((y^2, y^3, \ldots, y^k))$ would be ordered by $\succeq$ contradicting the fact that $h^{-1}((y^2, y^3, \ldots, y^k)) \subset B^+$. Thus $h^{-1}(\cdot)$ is a function. Continuity of this function is immediate.

Step 2. We now show that X is convex. By way of a contradiction suppose $a, b \in X$ and for some $\lambda \in (0,1)$ we have $\lambda a + (1-\lambda)b \notin X$. This implies that there is not some point $z \in B^+$ with $h(z) = \lambda a + (1-\lambda)b$. But since B is convex and $h^{-1}(a), h^{-1}(b) \in B^+$ (since $a, b \in X$) the point $w = \lambda h^{-1}(a) + (1-\lambda)h^{-1}(b) \in B$. But then the point

$$z' = \{\max_{z^1}(z^1, w^2, \ldots w^k) : (z^1, w^2, \ldots, w^k) \in B\} \tag{4}$$

is in B^+. By the definition of $h(\cdot)$, $h(z') = \lambda a + (1-\lambda)b$ and we have derived a contradiction.

Step 3. We now show that R_i^h is strictly convex for each $i \in N$. For arbitrary $i \in N$ and distinct $a, b \in X$, assume that $aR_i^h b$. By definition of R_i^h this means that $h^{-1}(a)R_i h^{-1}(b)$. For any $\lambda \in (0,1)$ the point $(\lambda h^{-1}(a) + (1-\lambda)h^{-1}(b)) \in B$ because B is convex. If this point is in B^+ the fact that $h^{-1}(a), h^{-1}(b) \in B^+$ and R_i is strictly convex implies that $(\lambda h^{-1}(a) + (1-\lambda)h^{-1}(b))P_i h^{-1}(b)$ and thus $(\lambda a + (1-\lambda)b)P_i^h b$ and we are finished. So suppose that $(\lambda h^{-1}(a) + (1-\lambda)h^{-1}(b)) \in B \backslash B^+$. By monotonicity of R_i this means that $h^{-1}(\lambda a + (1-\lambda b))P_i(\lambda h^{-1}(a) + (1-\lambda)h^{-1}(b))$ since $h^{-1}(\lambda a + (1-\lambda b)) \in B^+$ and it differs from $(\lambda h^{-1}(a) + (1-\lambda)h^{-1}(b))$ only in the first coordinate. Now since R_i is strictly convex for any $\lambda \in (0,1)$ we have $(\lambda h^{-1}(a) + (1-\lambda)h^{-1}(b))P_i h^{-1}(b)$. Transitivity then yields

$h^{-1}(\lambda a + (1-\lambda b))P_i h^{-1}(b)$ implying that $\lambda a + (1-\lambda)bP_i^h b$. Thus the relations R_i^h on X are strictly convex for each $i \in N$.

Step 4. Compactness of X is immediate as B^+ is compact (from lemma 4) and $h(\cdot)$ is continuous.

Step 5. We now show that R_i^h is continuous. For any $a \in X$, the fact that R_i is continuous implies that $P_i(h^{-1}(a))$ and $P_i^{-1}(h^{-1}(a))$ are open in $\mathbb{R}_+^k$. Now since h^{-1} is continuous the sets $h^{-1}(P_i(h^{-1}(a)))$ and $h^{-1}(P_i^{-1}(h^{-1}(a)))$ are relatively open in X. But by definition of R_i^h this means that $P_i^h(a)$ and $P_i^{h^{-1}}(a)$ are relatively open in X. ■

Figure 4 depicts a heuristic for this construction with $k=2$. The mapping $h(\cdot)$ involves flattening B^+ by equating the first coordinate of every point in B^+. This mapping is generally not invertible. But since every point in B^+ is efficient, given coordinates $(x^2, \ldots, x^k)$ of a point $x \in B^+$ the coordinate x^1 can be recovered by selecting the maximal value of x^1 s.t. $x \in B$.

Given lemma 4 the following equivalence is immediate.

Lemma 5. *If f is weakly Paretian then*

$$h(C(f,\rho,B)) = C(f,\rho^h,X)$$

and similarly

$$h^{-1}(C(f,\rho^h,X)) = C(f,\rho,B).$$

Since a homeomorphism preserves cardinality this Lemma and Proposition 1 imply that when B is compact and convex, the cardinality of $C(f,\rho,B)$ corresponds to that of $C(f,\rho^h,X)$ and this latter problem involves a compact and convex choice set with strictly convex and continuous preferences. The properties of this type of social choice problem are understood. For example if $\dim(B)=2$, then the problem $\langle N, X, \rho^h, f\rangle$ involves aggregation on the line with strictly convex preferences. Letting $a_i^* = \{x \in X : P_i^h(x) = \emptyset\}$ (which is non empty by compactness of X and continuity of R_i^h and a singleton by convexity of X and strict convexity of R_i^h) we define two set valued mappings:

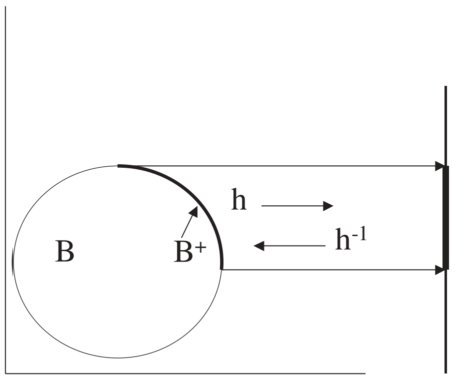

Fig. 4.

$L^+(z) = \{i \in N : a_i^* < z\}$ and $L^-(z) = \{i \in N : a_i^* > z\}$. For a policy $z \in X$ agents in $L^+(z)$ prefer a slightly higher policy, and agents in $L^-(z)$ prefer a slightly lower policy. The set of f−medians is then

$$\mu(f, \rho^h, X) = \{a \in X : L^-(z) \notin \mathscr{L}(f) \text{and } L^+(z) \notin \mathscr{L}(f)\}. \tag{5}$$

It is well known that the core $C(f, \rho^h, X)$ is equivalent to the set of f-medians $\mu(f, \rho^h, X)$. Thus we have the following characterization.

Corollary 1. *If* f *is weakly Paretian and* $\dim(B) = 2$ *then*

$$C(f, \rho, B) = h^{-1}(\mu(f, \rho^h, X)).$$

Proof. By proposition 1, ρ^h is a profile of strictly convex and continuous preferences on $X \subset \mathbb{R}^1_+$. Thus preferences are strictly single-peaked and the equivalence

$$C(f, \rho, B) = \mu(f, \rho^h, X) \tag{6}$$

follows from theorem 4.4 of Austen-Smith and Banks (1999). By lemma 5 the equivalence

$$h^{-1}(C(f, \rho^h, X)) = C(f, \rho, B) \tag{7}$$

attains. Thus, substitution leads to the desired result. ■

In the case of majority rule and n odd, proposition 2 yields a version of the median voter theorem on the efficient surface B^+. An immediate corollary is.

Corollary 2. *If* $\dim(B) = 2$, n *is odd and* f *is majority rule then* $C(f, \rho, B) = \{h^{-1}(a_m^*)\}$ *where*

$$m = \left\{ i \in N : \#\{j : a_j^* \leq a_i^*\} = \#\{j : a_j^* \geq a_i^*\} \right\}.$$

When $\dim(B) > 2$ the restriction of preferences afforded by the constraint and monotonicity results in a slight modification of the genericity results in Banks and Saari. These results assume that preferences are smooth and utilize singularity theory to characterize, for simple and q-rules, a bound on the dimensionality of the policy space for which it can be shown that generic profiles of smooth and strictly convex preferences will yield an empty core. Banks shows that if $k > 2(n - 2)$ then the core is generically empty. Saari shows that for a q-rule if $k > 2q - n$ then the core is generically empty. When preferences satisfy the additional assumption of strict monotonicity and choice is constrained by B the results must be relaxed.

Corollary 3. *If preferences are smooth and (1)* f *is weakly Paretian then* $C(f, \rho, B)$ *is generically empty if* $\dim(B) > 2(n - 2) + 1$; *(2) if* f *is a q-rule with* $\frac{n}{2} < q < n$ *then the core is generically empty if* $\dim(B) > 2q - n + 1$

Proof. (1) Since preferences are smooth, and the mapping $h : S^{\dim(B)} \to S^{\dim(B)-1}$ defined above is smooth, this mapping is a diffeomorphism. Thus, the Corollary to Theorem 2 in Banks (1995) applies to the problem with $\langle N, X, \rho^h, f \rangle$ which is a problem in $\mathbb{R}^{\dim(B)-1}$. Thus, $C(f, \rho^h, X)$ is generically empty if $\dim(B) - 1 > 2(n-2)$. (2) Similarly applying part a of Theorem 1 in Saari (1997) yields the conclusion that if f is a q-rule then (f, ρ^h, X) is generically empty if $\dim(B) - 1 > 2q - n$. ■

These corollaries are just suggestive uses of Proposition 1. More generally we have the following:

Theorem 1. (a) *If a proposition* Z *about the cardinality of* $C(f, \rho, X)$ *is true for a generic problem* $\langle N, X, \rho, f \rangle$ *with* $X \subset \mathbb{R}^k$ *compact and convex ,* ρ *strictly convex and continuous and* f *a weakly Paretian simple rule then proposition* Z *is true about* $C(f, \rho', B)$ *for a generic problem* $\langle N, B, \rho', f \rangle$ *where* $\rho' \in \mathscr{M}^n$ *and* B *is a compact and convex subset with dimension* $k+1$.

(b) *If a proposition* Z *about the cardinality of* $C(f, \rho, X)$ *is true for a generic problem* $\langle N, X, \rho, f \rangle$ *with* $X \subset \mathbb{R}^k$ *compact and* ρ *continuous and* f *a weakly Paretian simple rule then proposition* Z *is true about* $C(f, \rho', B)$ *for a generic problem* $\langle N, B, \rho', f \rangle$ where $\rho' \in \mathscr{M}^n$ *and* B *is a compact subset with dimension* $k+1$.

The theorem allows us to relate $\langle N, B, \rho, f \rangle$ to a lower dimensional problem with (and in the case of b, without) a convex choice set and strictly convex preferences if B is itself compact (and in the case of a) convex. The difference between parts a and b stems from the fact that Steps 3 and 4 in the proof of Proposition 1 hinge on convexity of B and it is the cases where B is not convex that are addressed by Lemma 4 (and thus part b of the proposition).

4 Discussion

Since Arrow's (1951) seminal work, scholars have considered the possibility of preference aggregation when preferences satisfy particular homogeneity conditions. Strict monotonicity is itself a homogeneity condition, requiring that all upper contour sets contain a translation of the positive othant Without a constraint the choice problem is trivial but not well defined as all agents would like to select an infinite quantity of each coordinate. With a compact constraint, the choice problem may not be trivial, but the effective dimensionality of the problem is reduced. Weakly Paretian rules will only choose an efficient policy, and the set of such policies has a dimensionality of one less than the constraint set. While the set of efficient and feasible policies may not be affine or convex, if the constraint set is itself convex it is possible to transform the space to make the efficient boundary convex without distorting any relevant features of the preferences.

While the homogeneity afforded by monotonicity and a constraint do not dramatically effect the results regarding core existence, for some problems the restriction is important. For example a problem of choosing government spending in two periods subject to a non linear intertemporal constraint or Downsian competition in which the government faces a feasibility constraint over social and domestic spending can be modeled in $\mathbb{R}^2_+$ with convex constraints that do not have affine efficient surfaces. The analysis here establishes that these models will have stable core points, whereas existing theorem's do no adjust for the homogeneity afforded by monotonicity and a compact constraint.

Throughout, we have maintained the assumption of strict monotonicity of preferences on $\mathbb{R}^k$. This condition is stronger than needed as preference homogeneity outside of B is of no relevance. Accordingly the condition can be relaxed to strict monotonicity on B. With Euclidean preferences, this condition is satisfied as long as all ideal points are exterior to (and to the north-east of) B.

References

Arrow K (1951) Social choice and individual values. Yale University Press, New Haven

Austen-Smith D, Banks J (1999) Positive political theory 1. The University of Michigan Press, Ann Arbor

Bredon G (1993) Topology and geometry. Springer, Berlin Heidelberg New York

Banks J (1995) Singularity theory and core existence in the spatial model. J Math Econ 24: 523–536

McKelvey R, Schofield N (1986) Structural instability of the core. J Math Econ 15: 179–198

McKelvey R, Schofield N (1987) Generalized symmetry conditions at a core point. Econometrica 55: 923–934

Plott C (1967) A notion of equilibrium and its possibility under majority rule. Am Econ Rev 57: 787–806

Saari D (1997) The generic existence of a core for q-rules. Econ Theory 9: 219–260

Zaller J (1998) Coming to grips with V.O. Key's concept of latent opinion. U.C.L.A. typescript