Nonnegative Polynomials in Optimization and Control

Amir Ali Ahmadi
Princeton University
Dept. of Operations Research and Financial Engineering (ORFE)

Dynamical Systems, Control and Optimization
Liège, Belgium
May 18, 2015
What is this talk about?

“Optimization over nonnegative polynomials”

1. Introduction to sum of squares (sos) programming
   Underlying numerical engine: SDP

2. “dsos and sdsos” programming
   Underlying numerical engine: LP/SOCP

Joint work with Anirudha Majumdar (MIT)
Optimization over Nonnegative Polynomials

**Defn.** A polynomial \( p(x) := p(x_1, \ldots, x_n) \) is nonnegative if \( p(x) \geq 0, \forall x \in \mathbb{R}^n \).

**Ex.** Decide if the following polynomial is nonnegative:

\[
p(x) = x_1^4 - 6x_1^3x_2 + 2x_1^3x_3 + 6x_1^2x_2^2 + 9x_1^2x_2^2 - 6x_1^2x_2x_3 - 14x_1x_2x_3^2 + 4x_1x_3^3 + 5x_3^4 - 7x_2^2x_3^2 + 16x_2^4
\]

**Basic semialgebraic set:**

\[
\{ x \in \mathbb{R}^n \mid f_i(x) \geq 0, h_i(x) = 0 \}
\]

**Ex.**

\[
2x_1 + 5x_1^2x_2 - x_3 \geq 0
\]
\[
5 - x_1^3 + 2x_1x_3 = 0
\]
Why would you want to do this?!

- Let’s start with four application areas...
1. Polynomial Optimization

\[
\min_x p(x) \\
\begin{align*}
  f_i(x) & \leq 0 \\
  h_i(x) & = 0
\end{align*}
\]

Decidable, but intractable
(includes your favorite NP-complete problem)

Equivalent formulation:

\[
\max_{\gamma} \gamma \\
p(x) - \gamma \geq 0
\]

\[\forall x \in \{ f_i(x) \leq 0, \ h_i(x) = 0 \}\]

- Many applications:
  - Combinatorial optimization
  - Option pricing with moment information
  - The optimal power flow (OPF) problem
  - Sensor network localization
PARTITION

**Input:** A list of positive integers $a_1, \ldots, a_n$.

**Question:** Can you split them into two bags such that the sum in one equals the sum in the other?

\[ \{5, 2, 1, 6, 3, 8, 5, 4, 1, 1, 10\} \quad \{5, 2, 1, 6, 3, 8, 5, 4, 1, 1, 10\} \]

- Note that the YES answer is easy to certify.
- How would you certify a NO answer?
2. Infeasibility Certificates in Discrete Optimization

**PARTITION**

**Input:** A list of positive integers $a_1, \ldots, a_n$.

**Question:** Can you split them into bags such that the sum in one equals the sum in the other?

$$[a_1, a_2, \ldots, a_n]$$

\[ \exists \, x_i \in \{-1, 1\} \text{ s.t. } \sum_{i=1}^{n} x_i a_i = 0 \]

Infeasible iff

\[ \sum_{i=1}^{n} \left( x_i^2 - 1 \right)^2 + \left( \sum_{i=1}^{n} x_i a_i \right)^2 > 0 \]
2. Discrete Optimization (Cont’d.)

- **How many final exams** can the President schedule on the same day at UCL, such that no student has to take more than one?

- **Nodes:** course numbers
- **Edges:** iff there is at least one student who is taking both courses

- Need the **independent set number** of the graph

\[ \alpha(G) = 9 \]
How to certify optimality?

- A theorem of Motzkin & Straus (1965):

\[
\alpha(G) \leq k
\]

if and only if

\[
-2k \sum_{(i,j) \in \overline{E}} x_i x_j y_i y_j - (1 - k) \left( \sum_{i=1}^{n} x_i^2 \right) \left( \sum_{i=1}^{n} y_i^2 \right)
\]

is nonnegative.

- Similar algebraic formulations for other combinatorial optimization problems...
3. Dynamical Systems & Trajectory Optimization

\[ \dot{x} = f(x) \]
\[ x_{k+1} = f(x_k) \]

Properties of interest:
- Stability of equilibrium points
- Boundedness of trajectories
- Invariance
- Safety, collision avoidance
- …

Dynamics of prices  Equilibrium populations  Spread of epidemics  Robotics
What does this have to do with optimization?

Questions about dynamical systems (e.g. stability, safety)

Lyapunov Theory

Search for functions satisfying certain properties (e.g. nonnegativity, convexity)

Ex. Lyapunov’s stability theorem.

Lyapunov function

\[ \dot{x} = f(x) \]

\[ V(x) : \mathbb{R}^n \rightarrow \mathbb{R} \]

\[ \dot{V}(x) = \langle \frac{\partial V}{\partial x}, f(x) \rangle \]

\[ V(x) > 0 \]

\[ -\dot{V}(x) > 0 \Rightarrow \text{GAS} \]

(similar local version)
4: Statistics and Machine Learning

- **Shape-constrained regression**
  - e.g., convex regression

\[ p(x) \text{ convex } \iff y^T \nabla^2 p(x) y \text{ psd} \]

- **Clustering with semialgebraic sets**

[AAA, Luss, Malioutov, ’14]
How would you prove nonnegativity?

**Ex.** Decide if the following polynomial is nonnegative:

\[ p(x) = x_1^4 - 6x_1^3x_2 + 2x_1^3x_3 + 6x_1^2x_3^2 + 9x_1^2x_2^2 - 6x_1^2x_2x_3 - 14x_1x_2x_3^2 + 4x_1x_3^3 + 5x_3^4 - 7x_2^2x_3^2 + 16x_2^4 \]

- Not so easy! (In fact, **NP-hard for degree ≥ 4**)

- But what if I told you:

\[ p(x) = (x_1^2 - 3x_1x_2 + x_1x_3 + 2x_3^2)^2 + (x_1x_3 - x_2x_3)^2 + (4x_2^2 - x_3^2)^2. \]

**Natural questions:**

- Is it any easier to test for a sum of squares (SOS) decomposition?
- Is every nonnegative polynomial SOS?
Is it any easier to decide sos?

- Yes! Can be reduced to a semidefinite program (SDP)
  - A broad generalization of linear programs
  - Can be solved efficiently (e.g., using interior point algorithms)

- Can also efficiently search and optimize over sos polynomials

- Numerous applications...

[Reference: Lasserre, Nesterov, Parrilo, Nesterov, Nemirovski, Alizadeh]
**Thm:** A polynomial \( p(x) \) of degree \( 2d \) is sos if and only if there exists a matrix \( Q \) such that

\[
Q \succeq 0, \\
p(x) = z(x)^T Q z(x),
\]

where \( z \) is the vector of monomials of degree up to \( d \)

\[
z = [1, x_1, x_2, \ldots, x_n, x_1 x_2, \ldots, x_n^d]^T
\]

The set of such matrices \( Q \) forms the feasible set of an SDP.
Example

\[ p(x) = x_1^4 - 6x_1^3x_2 + 2x_1^3x_3 + 6x_1^2x_2^2 + 9x_1^2x_2 - 6x_1^2x_2x_3 - 14x_1x_2x_3^2 + 4x_1x_3^3 + 5x_3^4 - 7x_2x_3^2 + 16x_2^4 \]

\[ p(x) = z^T Q z \]

\[ z = (x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2)^T \]

\[ Q = \begin{pmatrix}
1 & -3 & 0 & 1 & 0 & 2 \\
-3 & 9 & 0 & -3 & 0 & -6 \\
0 & 0 & 16 & 0 & 0 & -4 \\
1 & -3 & 0 & 2 & -1 & 2 \\
0 & 0 & 0 & -1 & 1 & 0 \\
2 & -6 & 4 & 2 & 0 & 5
\end{pmatrix} \]

\[ Q = \sum_{i=1}^{3} a_i a_i^T \]

\[ a_1 = (1, -3, 0, 1, 0, 2)^T, \quad a_2 = (0, 0, 0, 1, -1, 0)^T, \quad a_3 = (0, 0, 4, 0, 0, -1)^T \]

\[ p(x) = (x_1^2 - 3x_1x_2 + x_1x_3 + 2x_3^2)^2 + (x_1x_3 - x_2x_3)^2 + (4x_2^2 - x_3^2)^2. \]
Lyapunov theory with sum of squares (sos) techniques

Ex. Lyapunov’s stability theorem.

\[ \dot{x} = f(x) \]

Lyapunov function

\[ V(x) : \mathbb{R}^n \rightarrow \mathbb{R} \]

\[ \dot{V}(x) = \left< \frac{\partial V}{\partial x}, f(x) \right> \]

\[ V(x) \text{ sos } \]

\[ -\dot{V}(x) \text{ sos } \]

\[ \Rightarrow V(x) > 0 \]

\[ \Rightarrow -\dot{V}(x) > 0 \]

\( \Rightarrow \text{GAS} \)

(similar local version)
Global stability

\[ V(x) \text{ sos} \Rightarrow -\dot{V}(x) > 0 \Rightarrow \text{GAS} \]

Example.

\[
\begin{align*}
\dot{x}_1 &= -0.15x_1^7 + 200x_1^6x_2 - 10.5x_1^5x_2^2 - 807x_1^4x_2^3 + 14x_1^3x_2^4 + 600x_1^2x_2^5 - 3.5x_1x_2^6 + 9x_2^7 \\
\dot{x}_2 &= -9x_1^7 - 3.5x_1^6x_2 - 600x_1^5x_2^2 + 14x_1^4x_2^3 + 807x_1^3x_2^4 - 10.5x_1^2x_2^5 - 200x_1x_2^6 - 0.15x_2^7
\end{align*}
\]

Output of SDP solver:

\[
V = 0.02x_1^8 + 0.015x_1^7x_2 + 1.743x_1^6x_2^2 - 0.106x_1^5x_2^3 - 3.517x_1^4x_2^4 \\
+ 0.106x_1^3x_2^5 + 1.743x_1^2x_2^6 - 0.015x_1x_2^7 + 0.02x_2^8.
\]
Motzkin (1967): 

\[ M(x_1, x_2, x_3) = x_1^4 x_2^2 + x_1^2 x_2^4 - 3 x_1^2 x_2^2 x_3^2 + x_3^6 \]

Robinson (1973): 

\[ R(x_1, x_2, x_3, x_4) = x_1^2 (x_1 - x_4)^2 + x_2^2 (x_2 - x_4)^2 + x_3^2 (x_3 - x_4)^2 + 2 x_1 x_2 x_3 (x_1 + x_2 + x_3 - 2 x_4) \]
Failure of converse implications for Lyapunov analysis

\[\begin{align*}
\dot{x}_1 &= -x_1^3x_2^2 + 2x_1^3x_2 - x_1^3 + 4x_1^2x_2^2 - 8x_1^2x_2 + 4x_1^2 - x_1x_2^4 + 4x_1x_2^3 - 4x_1 + 10x_2^2 \\
\dot{x}_2 &= -9x_1^2x_2 + 10x_1^2 + 2x_1x_2^3 - 8x_1x_2^2 - 4x_1 - x_2^3 + 4x_2^2 - 4x_2
\end{align*}\]

[AAA, Parrilo]

- \(V(x) = x_1^2 + x_2^2\) proves GAS.
- SOS fails to find any quadratic Lyapunov function.

\[\begin{align*}
\dot{x} &= -x + xy \\
\dot{y} &= -y
\end{align*}\]

[AAA, Krstic, Parrilo]

- Globally asymptotically stable.
- But no polynomial Lyapunov function of any degree exists!
These examples are to be expected for complexity reasons

**Thm:** Deciding (local or global) asymptotic stability of cubic vector fields is strongly NP-hard.

[AAA]

**Implication:**

- Unless P=NP, there cannot be *any* polynomial time (or even pseudo-polynomial time) algorithm.

- In particular, the size of SOS certificates must be at least exponential.
Similar NP-hardness results for other problems

1. Inclusion of the unit ball in region of attraction \((d=3)\)
2. Invariance of the unit ball \((d=3)\)
3. Invariance of a quartic semialgebraic set \((d=1)\)
4. Boundedness of trajectories \((d=3)\)
5. Stability in the sense of Lyapunov \((d=4)\)
6. Local attractivity \((d=3)\)
7. Local collision avoidance \((d=4)\)
8. Existence of a quadratic Lyapunov function \((d=3)\)
9. Existence of a stabilizing control law \((d=3)\)
10. Local asymptotic stability for trigonometric vector fields \((d=4)\)
The good news

- In relatively small dimensions and degrees, it seems difficult to construct nonnegative polynomials that are not sos
- Especially true if additional structure is required
- For example, the following is OPEN:

Construct a *convex*, nonnegative polynomial that is not sos
(known to exist in high dimensions via a non-constructive proof of Blekherman)

- Empirical evidence from various domains over the last decade:

* SOS is a very powerful relaxation.*
Hilbert’s 17\textsuperscript{th} Problem (1900)

Q. \( p \) nonnegative \( \Rightarrow \ p = \sum \left( \frac{g_i}{q_i} \right)^2 \)

\textbullet Artin (1927): Yes!

\textbullet Implications:

\textbullet \( p \geq 0 \Rightarrow \exists h \text{ sos such that } p \cdot h \text{ sos} \)

\textbullet Reznick: (under mild conditions) can take \( h = (\sum_i x_i^2)^r \)

\textbullet Certificates of nonnegativity can \textit{always} be given with sos (i.e., with semidefinite programming)!

\textbullet We’ll see how the Positivstellensatz generalizes this even further...
Positivstellensatz: a complete algebraic proof system

- Let’s motivate it with a toy example:

Consider the task of proving the statement:

\[ \forall a, b, c, x, \ ax^2 + bx + c = 0 \Rightarrow b^2 - 4ac \geq 0 \]

Short algebraic proof (certificate):

\[ b^2 - 4ac = (2ax + b)^2 - 4a(ax^2 + bx + c) \]

- The Positivstellensatz vastly generalizes what happened here:
  - Algebraic certificates of infeasibility of any system of polynomial inequalities (or algebraic implications among them)
  - Automated proof system (via semidefinite programming)
Positivstellensatz: a generalization of Farkas lemma

Farkas lemma (1902):

\[ Ax = b \text{ and } x \geq 0 \text{ is infeasible} \]

\[ \iff \]

There exists a \( y \) such that \( y^T A \geq 0 \) and \( y^T b < 0 \).

(The S-lemma is also a theorem of this type for quadratics)
Positivstellensatz

Stengle (1974):

The basic semialgebraic set

\[ K := \{ x \in \mathbb{R}^n \mid g_i(x) \geq 0, \, i = 1, \ldots, m, \, h_i(x) = 0, \, i = 1, \ldots, k \} \]

is empty

\[ \uparrow \]

there exist polynomials \( t_1, \ldots, t_k \) and sum of squares polynomials \( s_0, s_1, \ldots, s_m, s_{12}, s_{13}, \ldots, s_{m-1m}, s_{123}, \ldots, s_{m-2m-1m}, \ldots, s_{12...m} \) such that

\[ -1 = \sum_{i=1}^{k} t_i(x)h_i(x) + s_0(x) + \sum_{\{i\}} s_i(x)g_i(x) \\
+ \sum_{\{i,j\}} s_{ij}(x)g_i(x)g_j(x) + \sum_{\{i,j,k\}} s_{ijk}(x)g_i(x)g_j(x)g_k(x) \\
+ \cdots + s_{ijk...m}(x)g_i(x)g_j(x)g_k(x) \cdots g_m(x). \]

Comments:

- Hilbert’s 17th problem is a straightforward corollary
- Other versions due to Shmudgen and Putinar (can look simpler)
Parrilo/Lasserre SDP hierarchies

Recall POP:
\[
\begin{align*}
\text{minimize} & \quad p(x) \\
\text{subject to} & \quad x \in K := \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, h_i(x) = 0\}
\end{align*}
\]

Idea:
obtain the largest lower bound by finding the largest \(\gamma\) for which the set \(\{x \in K, p(x) \leq \gamma\}\) is empty.
certify this emptiness by finding Positivstellensatz certificates.

\[
-1 = \sum_{i=1}^{k} t_i(x)h_i(x) + s_0(x) + \sum_{\{i\}} s_i(x)g_i(x) \\
+ \sum_{\{i,j\}} s_{ij}(x)g_i(x)g_j(x) + \sum_{\{i,j,k\}} s_{ijk}(x)g_i(x)g_j(x)g_k(x) \\
+ \cdots + s_{ijk\ldots m}(x)g_i(x)g_j(x)g_k(x)\cdots g_m(x)
\]

In level \(l\) of the hierarchy, degree of the polynomials \(t_i\) and the sos polynomials \(s_i\) is bounded by \(l\).

Comments:
- Each fixed level of the hierarchy is an SDP of polynomial size
- Originally, Parrilo’s hierarchy is based on Stengle’s Psatz, whereas Lasserre’s is based on Schmudgen’s Psatz
Local stability – SOS on the Acrobot

https://www.youtube.com/watch?v=FeCwtvrD76I

Swing-up:

Balance:

(4-state system)

Controller designed by SOS

[Majumdar, AAA, Tedrake ]

(Best paper award - IEEE Conf. on Robotics and Automation, ’13)
DSOS and SDSOS Optimization
Practical limitations of SOS

- **Scalability** is often a real challenge!!

**Thm:** \( p(x) \) of degree \( 2d \) is sos if and only if

\[
p(x) = z^T Q z \quad Q \succeq 0
\]

\[
z = [1, x_1, x_2, \ldots, x_n, x_1 x_2, \ldots, x_n^d]^T
\]

- The size of the Gram matrix is:

\[
\begin{pmatrix} n + d \end{pmatrix} \times \begin{pmatrix} n + d \end{pmatrix}
\]

- Polynomial in \( n \) for fixed \( d \), but grows quickly

  - **The semidefinite constraint is expensive**

- E.g., local stability analysis of a 20-state cubic vector field is typically an SDP with \(~1.2M\) decision variables and \(~200k\) constraints
Many interesting approaches to tackle this issue...

- Techniques for exploiting structure (e.g., symmetry and sparsity)
  - [Gatermann, Parrilo], [Vallentin], [de Klerk, Sotirov], [Papachristodoulou et al.], ...

- Customized algorithms (e.g., first order or parallel methods)
  - [Bertsimas, Freund, Sun], [Nie, Wang], [Peet et al.], ...

Our approach [AAA, Majumdar]:

- Let's not work with SOS to begin with...
- Give other sufficient conditions for non perhaps stronger than SOS, but hopefully cheaper
Not totally clear a priori how to do this...

Consider, e.g., the following two sets:

1) All polynomials that are **sums of 4\(^{th}\) powers of polynomials**
2) All polynomials that are **sums of 3 squares of polynomials**

Both sets are clearly inside the SOS cone

- But linear optimization over either set is **intractable**!
- So set inclusion doesn’t mean anything in terms of complexity
- We have to work a bit harder...
**dsos and sdsos**

**Defn.** A polynomial $p$ is *diagonally-dominant-sum-of-squares* (*dsos*) if it can be written as:

$$p = \sum_i \alpha_i m_i^2 + \sum_{i,j} \beta_{ij}^+ (m_i + m_j)^2 + \beta_{ij}^- (m_i - m_j)^2,$$

for some monomials $m_i, m_j$ and some nonnegative constants $\alpha_i, \beta_{i,j}$.

**Defn.** A polynomial $p$ is *scaled-diagonally-dominant-sum-of-squares* (*sdsos*) if it can be written as:

$$p = \sum_i \alpha_i m_i^2 + \sum_{i,j} (\beta_{i}^+ m_i + \gamma_{j}^+ m_j)^2 + (\beta_{i}^- m_i - \gamma_{j}^- m_j)^2,$$

for some monomials $m_i, m_j$ and some constants $\alpha_i \geq 0, \beta_i, \gamma_i$.

**Obvious:** $DSOS_{n,d} \subset SDSOS_{n,d} \subset SOS_{n,d} \subset POS_{n,d}$
**Defn.** A polynomial $p$ is *r-diagonally-dominant-sum-of-squares* (*r-dsos*) if 

$$p \cdot \left( \sum_i x_i^2 \right)^r$$

is dsos.

**Defn.** A polynomial $p$ is *r-scaled-diagonally-dominant-sum-of-squares* (*r-sdsos*) if 

$$p \cdot \left( \sum_i x_i^2 \right)^r$$

is sdsos.

Allows us to develop a **hierarchy** of relaxations...
dd and sdd matrices

**Defn.** A symmetric matrix \( A \) is *diagonally dominant* (**dd**) if

\[
a_{ii} \geq \sum_{j \neq i} |a_{ij}| \text{ for all } i.
\]

**Defn*. A symmetric matrix \( A \) is *scaled diagonally dominant* (**sdd**) if there exists a diagonal matrix \( D > 0 \) s.t.

\[
DAD \text{ is dd.}
\]

\[
\text{dd} \Rightarrow \text{sdd} \Rightarrow \text{psd}
\]

Greshgorin’s circle theorem

*Thanks to Pablo Parrilo for telling us about sdd matrices.*
$I + xA + yB$

$A, B$

$10 \times 10$

random

Optimization over these sets is an **SDP, SOCP, LP**!!
Two natural matrix programs: DDP and SDPP

**LP:**
\[
\begin{align*}
\min \langle C, X \rangle \\
A(X) &= b \\
X &\text{ diagonal\&nonnegative}
\end{align*}
\]

**DDP:**
\[
\begin{align*}
\min \langle C, X \rangle \\
A(X) &= b \\
X &\text{ dd}
\end{align*}
\]

**SDDP:**
\[
\begin{align*}
\min \langle C, X \rangle \\
A(X) &= b \\
X &\text{ sdd}
\end{align*}
\]

**SDP:**
\[
\begin{align*}
\min \langle C, X \rangle \\
A(X) &= b \\
X &\succeq 0
\end{align*}
\]
From matrices to polynomials

**Thm.** A polynomial $p$ is **dsos**

\[ p = \sum_i \alpha_i m_i^2 + \sum_{i,j} \beta_{ij}^+ (m_i + m_j)^2 + \beta_{ij}^- (m_i - m_j)^2, \]

if and only if

\[ p(x) = z^T(x)Qz(x) \]

\[ Q \text{ sdd} \]

**Thm.** A polynomial $p$ is **sdsos**

\[ p = \sum_i \alpha_i m_i^2 + \sum_{i,j} (\beta_{ij}^+ m_i + \gamma_{ij}^+ m_j)^2 + (\beta_{ij}^- m_i - \gamma_{ij}^- m_j)^2, \]

if and only if

\[ p(x) = z^T(x)Qz(x) \]

\[ Q \text{ sdd} \]
Optimization over r-dsos and r-dsos polynomials

• Can be done by LP and SOCP respectively!
• Commercial solvers such as CPLEX and GUROBI are very mature (very fast, deal with numerical issues)
• iSOS: add-on to SPOTless (package by Megretski, Tobenkin, Permenter –MIT)

https://github.com/spot-toolbox/spotless

How well does it do?!

• We show encouraging experiments from:
  Control, polynomial optimization, statistics, copositive programming, combinatorial optimization, options pricing, sparse PCA, etc.
• And we’ll give Positivstellensatz results (converse results)
First observation: $r$-dsos can outperform sos

The Motzkin polynomial:

$$M(x_1, x_2, x_3) = x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_1^2 x_2^2 x_3^2 + x_3^6$$

psd but not sos!

...but it’s 2-dsos.

(certificate of nonnegativity using LP)

Another ternary sextic:

$$p(x_1, x_2, x_3) = x_1^4 x_2^2 + x_2^4 x_3^2 + x_3^4 x_1^2 - 3x_1^2 x_2^2 x_3^2$$

not sos but 1-dsos (hence psd)
A parametric family of polynomials

\[ p(x) = 2x_1^4 + cx_2^4 + ax_1^2x_2^2 + bx_1^3x_2 \]

Compactify:

\[ p(x) = 2x_1^4 + (8 - a - b)x_2^4 + ax_1^2x_2^2 + bx_1^3x_2 \]
Minimizing a form on the sphere

\[
\min_{x \in S^{n-1}} p(x)
\]

- degree=4; all coefficients present – generated randomly

<table>
<thead>
<tr>
<th>n=10</th>
<th>Lower bound</th>
<th>Run time (secs)</th>
<th>n=15</th>
<th>Lower bound</th>
<th>Run time (secs)</th>
<th>n=20</th>
<th>Lower bound</th>
<th>Run time (secs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SOS</td>
<td>-1.920</td>
<td>1.01</td>
<td>SOS</td>
<td>-3.263</td>
<td>165.3</td>
<td>SOS</td>
<td>-3.579</td>
<td>5749</td>
</tr>
<tr>
<td>mosek</td>
<td>-1.920</td>
<td>0.184</td>
<td>mosek</td>
<td>-3.263</td>
<td>5.537</td>
<td>mosek</td>
<td>-3.579</td>
<td>79.06</td>
</tr>
<tr>
<td>sdsos</td>
<td>-5.046</td>
<td>0.152</td>
<td>sdsos</td>
<td>-10.433</td>
<td>0.444</td>
<td>sdsos</td>
<td>-17.333</td>
<td>1.935</td>
</tr>
<tr>
<td>dsos</td>
<td>-5.312</td>
<td>0.067</td>
<td>dsos</td>
<td>-10.957</td>
<td>0.370</td>
<td>dsos</td>
<td>-18.015</td>
<td>1.301</td>
</tr>
<tr>
<td>BARON</td>
<td>-175.4</td>
<td>0.35</td>
<td>BARON</td>
<td>-1079.9</td>
<td>0.62</td>
<td>BARON</td>
<td>-5287.9</td>
<td>3.69</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>n=30</th>
<th>Lower bound</th>
<th>Run time (secs)</th>
<th>n=40</th>
<th>Lower bound</th>
<th>Run time (secs)</th>
<th>n=50</th>
<th>Lower bound</th>
<th>Run time (secs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SOS</td>
<td>----------</td>
<td>(\infty)</td>
<td>SOS</td>
<td>----------</td>
<td>(\infty)</td>
<td>SOS</td>
<td>----------</td>
<td>(\infty)</td>
</tr>
<tr>
<td>mosek</td>
<td>----------</td>
<td>(\infty)</td>
<td>mosek</td>
<td>----------</td>
<td>(\infty)</td>
<td>mosek</td>
<td>----------</td>
<td>(\infty)</td>
</tr>
<tr>
<td>sdsos</td>
<td>-36.038</td>
<td>9.431</td>
<td>sdsos</td>
<td>-61.248</td>
<td>53.95</td>
<td>sdsos</td>
<td>-93.22</td>
<td>100.5</td>
</tr>
<tr>
<td>dsos</td>
<td>-36.850</td>
<td>8.256</td>
<td>dsos</td>
<td>-62.2954</td>
<td>26.02</td>
<td>dsos</td>
<td>-94.25</td>
<td>72.79</td>
</tr>
<tr>
<td>BARON</td>
<td>-28546.1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

PC: 3.4 GHz, 16 Gb RAM
Convex regression

300 points in $\mathbb{R}^{20}$

Observation: $e \|x\| + \text{noise}$

Best convex polynomial fit of degree $d$

(sd)sos constraint in 40 variables:

\[
y^T H(x) y \quad (sd)sos
\]

<table>
<thead>
<tr>
<th>$d=2$</th>
<th>Max Error</th>
<th>Run time (secs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SOS (mosek)</td>
<td>21.282</td>
<td>$\sim 1$</td>
</tr>
<tr>
<td>sdsos</td>
<td>33.918</td>
<td>$\sim 1$</td>
</tr>
<tr>
<td>dsos</td>
<td>35.108</td>
<td>$\sim 1$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$d=4$</th>
<th>Max Error</th>
<th>Run time (secs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SOS (mosek)</td>
<td>-------</td>
<td>$\infty$</td>
</tr>
<tr>
<td>sdsos</td>
<td>12.936</td>
<td>231</td>
</tr>
<tr>
<td>dsos</td>
<td>14.859</td>
<td>150</td>
</tr>
</tbody>
</table>
Some control applications
Stabilizing the inverted N-link pendulum (2N states)

Runtime:

<table>
<thead>
<tr>
<th>2N (# states)</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
<th>18</th>
<th>20</th>
<th>22</th>
</tr>
</thead>
<tbody>
<tr>
<td>DSOS</td>
<td>&lt; 1</td>
<td>0.44</td>
<td>2.04</td>
<td>3.08</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SDSOS</td>
<td>&lt; 1</td>
<td>0.72</td>
<td>6.72</td>
<td>7.78</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SOS (SeDuMi)</td>
<td>&lt; 1</td>
<td>3.97</td>
<td>156.9</td>
<td>1697.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SOS (MOSEK)</td>
<td>&lt; 1</td>
<td>0.84</td>
<td>16.2</td>
<td>149.1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

ROA volume ratio:

<table>
<thead>
<tr>
<th>2N (states)</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_{dsos}/\rho_{sos}$</td>
<td>0.38</td>
<td>0.45</td>
<td>0.13</td>
<td>0.12</td>
<td>0.09</td>
</tr>
<tr>
<td>$\rho_{adsos}/\rho_{sos}$</td>
<td>0.88</td>
<td>0.84</td>
<td>0.81</td>
<td>0.79</td>
<td>0.79</td>
</tr>
</tbody>
</table>

(w/ Majumdar, Tedrake)
Stabilizing ATLAS

- 30 states
- 14 control inputs
- Cubic dynamics

https://www.youtube.com/watch?v=lmAT556Ar5c

Done by SDSOS Optimization

[Majumdar, AAA, Tedrake]
Lyapunov Barrier Certificates

\[ \dot{x} = f(x) \]

(vector valued polynomial)

\( S \): needs safety verification

\( U \): unsafe (or forbidden) set

(both sets semialgebraic)

Safety assured if we find a “Lyapunov function” such that:

\[
\begin{align*}
B(S) &< 0 \\
B(U) &> 0
\end{align*}
\]

\[
\dot{B} = \left\langle \nabla B(x), f(x) \right\rangle \leq 0
\]
Real-time collision avoidance

Done by SDSOS Optimization

Dubins car model

Run-time: 20 ms

https://www.youtube.com/watch?v=J3a6v0tIsD4

(w/ Majumdar)
Converse results 1&2

**Thm.** Any even positive definite form \( p \) is \( r \)-dsos for some \( r \).

- Hence proof of positivity can always be found with LP
- Proof follows from a result of Polyá (1928) on Hilbert’s 17th problem
- Even forms include, e.g., copositive programming!
- \( r \leq \alpha^2(G) \) for finding independent sets in graphs
  (corollary of a result of de Klerk & Pasechnik)

**Thm.** Any positive definite bivariate form \( p \) is \( r \)-sdsos for some \( r \).

- Proof follows from a result of Reznick (1995)
  - \( p.||x||^{r} \) will always become a sum of powers of linear forms for sufficiently large \( r \).
Converse result 3 & Polynomial Optimization

**Thm.** For any positive definite form $p$, there exists an integer $r$ and a polynomial $q$ of degree $r$ such that

$$q \text{ is dsos and } pq \text{ is dsos}.$$

- Search for $q$ is an LP
- Such a $q$ is a certificate of nonnegativity of $p$
- Proof follows from a result of Habicht (1940) on Hilbert’s 17th problem

- Similar to the Lasserre/Parrilo SDP hierarchies, polynomial optimization can be solved to global optimality using hierarchies of LP and SOCP coming from dsos and sdsos.
Ongoing directions...

Iterative DSOS via
- Column generation
- Cholesky change of basis

(w/ S. Dash, IBM)
Main messages...

Many applications

Cool and classical theory
Rejuvenated thanks to modern interface with computation and optimization

Move from SDP to LP and SOCP
New possibilities likely to become within reach

Want to know more? aaa.princeton.edu


- A.A. Ahmadi and A. Majumdar, Some applications of polynomial optimization in operations research and real-time optimization, Optimization Letters, accepted for publication, 2014.

- A. Majumdar, A.A. Ahmadi, and R. Tedrake, Control and verification of high-dimensional systems with DSOS and SDSOS optimization, CDC 2014.


- A.A. Ahmadi, A. Majumdar, and R. Tedrake, Complexity of ten decision problems in continuous time dynamical systems, ACC 2013.

- A.A. Ahmadi, M. Krstic, and P.A. Parrilo, A globally asymptotically stable polynomial vector field with no polynomial Lyapunov function, CDC 2012.

- A.A. Ahmadi and P.A. Parrilo, Converse results on existence of sum of squares Lyapunov functions, CDC 2012.