Complexity of Testing Existence of Solutions in Polynomial Optimization

+ A New Positivstellensatz

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Existence of solutions

Consider a polynomial optimization problem (POP):

\[
\inf_{x \in \mathbb{R}^n} p(x) \\
\text{s.t. } g_i(x) \geq 0, i = 1, \ldots, m.
\]

Suppose the optimal value is finite (i.e., POP is feasible and bounded below). We would like to test if there exists an optimal solution, i.e., a feasible point \(x^*\) such that \(p(x^*) \leq p(y), \forall y \text{ feasible.}\)

Informally: “Can we replace the ‘inf’ with a ‘min’?”

Remarks:

• If feasible set is bounded, a solution always exists.
• If \(n = 1\), a solution always exists.
• Finiteness of optimal value comes as a “promise”.
Motivation

- An exact algorithm cannot return a solution if there is none!
- Existence of solutions essential for algorithms that exploit optimality conditions.

[Nie, Demmel, Sturmfels, “Minimizing polynomials via sum of squares over the gradient ideal”, Math. Prog. 2005]:

“This assumption [existence of minimizers] is nontrivial, and we do not address the (important and difficult) question of how to verify that a given polynomial \( p(x) \) has this property.”

There are algorithms that check existence of solutions:

- Greuet, Safey El Din, “Probabilistic algorithm for polynomial optimization over a real algebraic set”, *SIAM J. on Optimization*, 2014
- Quantifier elimination
- …

*All have running time at least exponential in dimension… Can there be a faster algorithm?*
Existence of a solution guaranteed?

\[
\min_{x \in \mathbb{R}^n} p(x) \\
\text{s.t. } g_i(x) \geq 0, \; i = 1, \ldots, m
\]

Degree of constraints

<table>
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<th>Degree of objective</th>
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<th>2</th>
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<tr>
<td>1</td>
<td>Yes</td>
<td>Yes Linear Programming</td>
<td>NP-hard to test (This work)</td>
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<tr>
<td>2</td>
<td>Yes</td>
<td>Linear Algebra Frank, Wolfe (1956)</td>
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<tr>
<td>3</td>
<td>Yes</td>
<td>Andronov, Belousov, Shironin (1982)</td>
<td></td>
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<tr>
<td>4</td>
<td>NP-hard to test (This work)</td>
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</table>
1) NP-hardness of testing existence of solutions

2) Sufficient conditions for existence of solutions
   a. Review of SOS and Positivstellensätze
   b. An SOS hierarchy for coercivity

3) An optimization-free Positivstellensatz
   (brief and independent)
Outline

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Main hardness results

Theorem (AAA, Zhang)  

Testing whether a degree-4 polynomial attains its unconstrained infimum is strongly NP-hard.

Theorem (AAA, Zhang)  

Testing whether a degree-1 polynomial attains its infimum on a set defined by degree-2 inequalities is strongly NP-hard.

Proof: Reduction from 1-in-3 3SAT.
1-in-3 3SAT

- **Input:** A CNF formula with three literals per clause.
- **Goal:** Find a Boolean assignment so that each clause has exactly one true literal.

\[
\begin{align*}
&1 \quad -1 \quad -1 \quad -1 \quad 1 \quad -1 \quad -1 \quad 1 \\
&(x_1 \lor \neg x_2 \lor x_3) \land (\neg x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2 \lor \neg x_3) \\
&x_1 = 1, x_2 = 1, x_3 = -1
\end{align*}
\]

\[
(x_1 \lor \neg x_2 \lor x_3) \land (\neg x_1 \lor x_2 \lor x_3) \land (\neg x_1 \lor \neg x_2 \lor x_3)
\]

Not satisfiable.

This problem is NP-hard.
NP-hardness of checking attainment (1/2)

**Goal:** Given any instance of 1-in-3 3SAT, construct a polynomial that attains its infimum if and only if the instance is satisfiable

**Step 1:**

\[
\phi = (x_1 \lor x_2 \lor \neg x_3) \land (\neg x_1 \lor \neg x_2 \lor x_3)
\]

\[
p_\phi(x) := \sum_{i=1}^{n} (1 - x_i^2)^2 + (x_1 + x_2 - x_3 + 1)^2 + (-x_1 - x_2 + x_3 + 1)^2
\]

**Important Property:** \(p_\phi(x)\) has a zero if and only if \(\phi\) is satisfiable

But \(p_\phi(x)\) **always attains its infimum** (independent of whether \(\phi\) is satisfiable) as its highest order component is \(\sum_{i=1}^{n} x_i^4\).
NP-hardness of checking attainment (2/2)

Step 2:

\[ q_\phi(x_1, \ldots, x_n, y, z, \lambda) = \]

\[(1 - \lambda)^2(y^2 + (1 - yz)^2) + \lambda^2 p_\phi(x) \]

- Infimum is always zero (\(q_\phi\) is a sum of squares; take \(\lambda = 0, y \to 0, z = \frac{1}{y}\)).
- If \(\phi\) satisfiable, take \(\lambda = 1\), and \(x\) the satisfying assignment \(\Rightarrow\) Infimum attained.
- If \(\phi\) not satisfiable, \(q_\phi\) does not vanish \(\Rightarrow\) Infimum not attained.
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Sufficient conditions for existence of solutions

\[
\min_{x \in \mathbb{R}^n} p(x) \\
\text{s.t. } g_i(x) \geq 0, \ i = 1, \ldots, m
\]

- Stable Compactness
- Coercivity of the objective function
- Convexity of the objective function and concavity of the constraints
- Compactness of the feasible set
- Archimedean Condition

**Theorem**

- Testing whether a polynomial optimization problem satisfies any of these conditions is strongly NP-hard.
- Our results are minimal in the degree.
Review of sum of squares and Positivstellensatze
How to prove positivity?

Is \( p(x) > 0 \) on \( \{g_1(x) \geq 0, \ldots, g_m(x) \geq 0\} \)?

Why prove positivity?

- **Infeasibility certificates** for systems of polynomial inequalities
  \[
  \{g_1(x) \geq 0, g_2(x) \geq 0, \ldots, g_m(x) \geq 0\} \text{ empty} \iff \\
  -g_1(x) > 0 \text{ on } \{g_2(x) \geq 0, \ldots, g_m(x) \geq 0\}
  \]

- (Tight) lower bounds for polynomial minimization problems
- **Dynamics and control** (Lyapunov functions)
- **Stats/ML** (shape-constrained regression),...
Sum of squares and SDP

- A polynomial \( p \) is a sum of squares (sos) if it can be written as

\[
p(x) = \sum_i q_i^2(x),
\]

where \( q_i \) are polynomials.

Ex: \[
p(x) = x_1^4 - 6x_1^3x_2 + 2x_1^3x_3 + 6x_1^2x_2^2 + 9x_1^2x_2^2 - 6x_1^2x_2x_3 - 14x_1x_2x_3^2 + 4x_1x_3^3 + 5x_3^4 - 7x_2^2x_3^2 + 16x_2^4
\]

\[
= (x_1^2 - 3x_1x_2 + x_1x_3 + 2x_3^2)^2 + (x_1x_3 - x_2x_3)^2 + (4x_2^2 - x_3^2)^2
\]

- A polynomial \( p \) of degree \( 2d \) is sos if and only if \( \exists Q \succeq 0 \) such that

\[
p(x) = z(x)^T Q z(x)
\]

where \( z = [1, x_1, ..., x_n, x_1x_2, ..., x_n^d]^T \) is the vector of monomials of degree up to \( d \).

Optimizing over set of sos polynomials is an SDP!
If $p(x) > 0$, then $\exists$ sos $q$ s.t. $p \cdot q$ sos.

If $p(x) \geq 0$, then $\exists$ sos $q$ s.t. $p \cdot q$ sos.

If $p(x) > 0$, $\forall x \in S = \{x \mid g_i(x) \geq 0\}$, then $p(x) = \sigma_0(x) + \sum_i \sigma_i(x)g_i(x) + \sum_{ij} \sigma_{ij}(x)g_i(x)g_j(x) + \cdots$, where $\sigma_0, \sigma_i, \ldots$ sos

Requires compactness.

If $p(x) > 0$, $\forall x \in S$, then $p(x) = \sigma_0(x) + \sum_i \sigma_i(x)g_i(x)$, where $\sigma_0, \sigma_i$ are sos.

Requires the Archimedean property.

Search for these sos polynomials (when degree is fixed) --->SDP.
### Infeasibility proofs for polynomial (in)equalities

**Stengle’s Positivstellensatz**

\[
\{ \{ g_i(x) \geq 0, i = 1, \ldots, m, h_i(x) = 0, i = 1, \ldots, t \} \text{ infeasible} \}
\]

if and only if

there exist polynomials \( \tau_i \) and sos polynomials \( \sigma \) such that

\[
-1 = \sum \tau_i h_i + \sigma_0 + \sum \sigma_i g_i + \sum \sigma_{ij} g_i g_j + \sum \sigma_{ijk} g_i g_j g_k \\
+ \ldots + \sigma_{1 \ldots m} \prod g_1 \ldots g_m.
\]

Search for these sos certificates of infeasibility (when deg. is fixed) ---> SDP.
Back to coercivity
Coercivity

**Definition:** A function $f$ is *coercive* if for every sequence $\{x_k\}$ such that $\|x_k\| \to \infty$, we have $f(x) \to \infty$.

- A coercive function attains its infimum
- Checking whether a polynomial is coercive is NP-hard

**Past work:**
- Jeyakumar, Lasserre (2014)
  - SDP hierarchy
- Bajbar, Stein (2015)
- Bajbar, Behrends (2017)

We provide a condition which is (i) both necessary and sufficient for a polynomial to be coercive and (ii) amenable to an SDP hierarchy
An sos hierarchy for testing coercivity (1/2)

Theorem (AAA, Zhang)

A polynomial $p$ of degree $d$ is coercive if and only if for some integer $r \geq 1$ the following SDP is feasible

\[-1 = \sigma_0(x, \gamma) + \sigma_1(x, \gamma)(\gamma - p(x)) + \sigma_2(x, \gamma) \left( \sum_{i=1}^{n} x_i^2 - \gamma^{2r} - 2^r \right) + \sigma_3(x, \gamma)(\gamma - p(x)) \left( \sum_{i=1}^{n} x_i^2 - \gamma^{2r} - 2^r \right),\]

- $\sigma_0$ is sos and of degree $\leq 4r$,
- $\sigma_1$ is sos and of degree $\leq \max\{4r - d, 0\}$,
- $\sigma_2$ is sos and of degree $\leq 2r$,
- $\sigma_3$ is sos and of degree $\leq \max\{2r - d, 0\}$.
A function is coercive if and only if all its sublevel sets are compact.

**Theorem (AAA, Zhang)**

A polynomial $p$ is coercive if and only if there exist an even integer $c > 0$ and a scalar $k \geq 0$ such that for all $\gamma \in \mathbb{R}$, the $\gamma$-sublevel set of $p$ is contained within a ball of radius $\gamma^c + k$.

$$p(x) = x^4 + y^4 - 2x^3 + y^2 + 3x + y$$

Equivalently, $p$ is coercive if and only if there exist an even integer $c'$ and a scalar $k'$ for which the set $\{(x, \gamma)|p(x) \leq \gamma, \Sigma x_i^2 \geq \gamma^{c'} + k'\}$ is empty.
Theorem (AAA, Zhang)

A polynomial \( p \) of degree \( d \) is coercive if and only if for some integer \( r \geq 1 \) the following SDP is feasible:

\[
-1 = \sigma_0(x, \gamma) + \sigma_1(x, \gamma)(\gamma - p(x)) + \sigma_2(x, \gamma) \left( \sum_{i=1}^{n} x_i^2 - \gamma^{2r} - 2^r \right) \\
+ \sigma_3(x, \gamma)(\gamma - p(x)) \left( \sum_{i=1}^{n} x_i^2 - \gamma^{2r} - 2^r \right),
\]

- \( \sigma_0 \) is sos and of degree \( \leq 4r \),
- \( \sigma_1 \) is sos and of degree \( \leq \max\{4r - d, 0\} \),
- \( \sigma_2 \) is sos and of degree \( \leq 2r \),
- \( \sigma_3 \) is sos and of degree \( \leq \max\{2r - d, 0\} \).
Toy example

The polynomial \( p(x) = x_1^4 + x_2^2 \) is coercive as certified by the following algebraic identity:

\[
-1 = \left( \frac{2}{3} \left( x_1^2 - \frac{1}{2} \right)^2 + \frac{2}{3} \left( \gamma - \frac{1}{2} \right)^2 \right) + \frac{2}{3} (\gamma - x_1^4 - x_2^2) + \frac{2}{3} (x_1^2 + x_2^2 - \gamma^2 - 2)
\]
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   (brief and independent)
Recall what a Positivstellensatz establishes

\[ p(x) > 0, \forall x \in \{ x \in \mathbb{R}^n \mid g_i(x) \geq 0, i = 1, \ldots, m \} \]

Under the Archimedean property

If \( p(x) > 0, \forall x \in S, \)
then \( p(x) = \sigma_0(x) + \sum_i \sigma_i(x)g_i(x), \)
where \( \sigma_0, \sigma_i \) are sos

Search for these sos polynomials (when degree is fixed) \( \rightarrow \) SDP.
Similar situation for Psatze of Stengle and Schmudgen.
An optimization-free Positivstellensatz (1/2)

\[ p(x) > 0, \forall x \in \{ x \in \mathbb{R}^n \mid g_i(x) \geq 0, i = 1, \ldots, m \} \]

\[ 2d = \text{maximum degree of } p, g_i \]

\[ \Leftrightarrow \exists r \in \mathbb{N} \text{ such that} \]

\[ \left( f(v^2 - w^2) - \frac{1}{r} (\sum_i (v_i^2 - w_i^2)^2)^d + \frac{1}{2r} (\sum_i (v_i^4 + w_i^4))^d \right) \cdot (\sum_i v_i^2 + \sum_i w_i^2)^{r^2} \]

has nonnegative coefficients,

where \( f \) is a form in \( n + m + 3 \) variables and of degree \( 4d \), which can be explicitly written from \( p, g_i \) and \( R \).
An optimization-free Positivstellensatz (2/2)

\[ p(x) > 0 \text{ on } \{x \mid g_i(x) \geq 0\} \iff \exists r \in \mathbb{N} \text{ s.t.} \left( f(v^2 - w^2) - \frac{1}{r}(\sum_i(v_i^2 - w_i^2)^2)^d + \frac{1}{2r}(\sum_i(v_i^4 + w_i^4))^d \right) \cdot (\sum_i v_i^2 + \sum_i w_i^2)^r \]

has \( \geq 0 \) coefficients

- \( p(x) > 0 \text{ on } \{x \mid g_i(x) \geq 0\} \iff f \text{ is pd} \)
- **Result by Polya (1928):**
  \( f \) **even** and pd \( \Rightarrow \exists r \in \mathbb{N} \) such that \( f(z) \cdot (\sum_i z_i^2)^r \) has nonnegative coefficients.
  - Make \( f(z) \) even by considering \( f(v^2 - w^2) \). We lose positive definiteness of \( f \) with this transformation.
  - Add the positive definite term \( \frac{1}{2r}(\sum_i(v_i^4 + w_i^4))^d \) to \( f(v^2 - w^2) \) to make it positive definite. **Apply Polya’s result.**
  - The term \( -\frac{1}{r} \left( \sum_i(v_i^2 - w_i^2)^2 \right)^d \) ensures that the converse holds as well.

Want to know more? [aaa.princeton.edu](aaa.princeton.edu)
You are cordially invited...

Princeton Day of Optimization
September 28, 2018
http://orfe.princeton.edu/pdo/

Thank you.