Dimension reduction for semidefinite programming

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Semidefinite programs (SDPs)

\[
\begin{align*}
\text{minimize} & \quad \text{Tr } CX \\
\text{subject to} & \quad X \in A \cap S_n^+ \\
\end{align*}
\]

Formulated over vector space $S^n$ of $n \times n$ symmetric matrices.

- variable $X \in S^n$
- $A \subseteq S^n$ an affine subspace, $C \in S^n$ cost matrix
- $S_n^+$ cone of psd matrices

Efficiently solvable in theory; in practice, solving some instances impossible unless special structure is exploited.
Dimension reduction

Reformulate problem over subspace $S \subseteq \mathbb{S}^n$ intersecting set of optimal solutions

$$\begin{align*}
\text{minimize} & \quad \text{Tr } CX \\
\text{subject to} & \quad X \in A \cap S^n
\end{align*}$$

$$\begin{align*}
\text{minimize} & \quad \text{Tr } CX \\
\text{subject to} & \quad X \in A \cap S^n \cap S
\end{align*}$$

(Reformulation)

where $S^n_+ \cap S$ equals product $\mathcal{K}_i \times \cdots \times \mathcal{K}_m$ of ‘simple’ cones.

Reduction methods: *symmetry reduction* and *facial reduction*
Symmetry reduction (MAXCUT relaxation example)

minimize $\text{Tr } CX$

subject to $X \in A \cap S^n$

$A := \{ X \in S^n : X_{ii} = 1 \}$

$C := \text{adjacency matrix}$
Symmetry reduction (MAXCUT relaxation example)

\[ \text{minimize} \quad \text{Tr} \ CX \]
\[ \text{subject to} \quad X \in \mathcal{A} \cap \mathbb{S}^n_+ \]

\[ \mathcal{A} := \{ X \in \mathbb{S}^n : X_{ii} = 1 \} \]
\[ C := \text{adjacency matrix} \]

Idea: find special projection map \( P \)
- \( P(X) \) optimal when \( X \) optimal.
- \( P \) explicitly constructed from automorphism group of graph.
- Range 'block-diagonal'—a direct-sum of matrix algebras.

(e.g., Schrijver '79; Gatermann-P. '03)
Facial reduction

First, find face of $S^n_+$ containing feasible set.
- There exists a hyperplane $H^\perp$ containing $\mathcal{A}$.
- $S^n_+ \cap H^\perp$ a face—isomorphic to $S^d_+$ for $d < n$.
- Face $S^n_+ \cap H^\perp$ contains feasible set $\mathcal{A} \cap S^n_+$.

minimize $\operatorname{Tr} CX$
subject to $X \in \mathcal{A} \cap S^n_+$
Facial reduction

First, find face of $S^n_+$ containing feasible set.
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- $S^n_+ \cap H^\perp$ is a face—isomorphic to $S^d_+$ for $d < n$.
- Face $S^n_+ \cap H^\perp$ contains feasible set $\mathcal{A} \cap S^n_+$.

Next, reformulate SDP over face:

$$\begin{align*}
\text{minimize} & \quad \text{Tr } CX \\
\text{subject to} & \quad X \in \mathcal{A} \cap S^n_+ \cap H^\perp
\end{align*}$$
Facial reduction

First, find face of $\mathbb{S}_+^n$ containing feasible set.

- There exists a hyperplane $H^\perp$ containing $\mathcal{A}$.
- $\mathbb{S}_+^n \cap H^\perp$ a face—isomorphic to $\mathbb{S}_+^d$ for $d < n$.
- Face $\mathbb{S}_+^n \cap H^\perp$ contains feasible set $\mathcal{A} \cap \mathbb{S}_+^n$.

Next, reformulate SDP over face:

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\text{minimize} & \quad \text{Tr } CX \\
\text{subject to} & \quad X \in \mathcal{A} \cap \mathbb{S}_+^n \cap H^\perp
\end{align*}$$

Borwein-Wolkowicz ’81; Pataki ’00; Permenter-P. ’14
Application specific approaches

Facial reduction:
- MAXCUT (Anjos, Wolkowicz)
- QAP (Zhao, Wolkowicz)
- Sums-of-squares optimization (Permenter-P., Waki-Muramatsu)
- Matrix completion (Krislock, Wolkowicz)
- ...

Symmetry reduction:
- MAXCUT (earlier example),
- QAP (de Klerk, Sotirov);
- Markov chains (Boyd et al.);
- codes (Schrijver; Laurent)
- ...

Our approach

This talk: new reduction method subsuming symmetry reduction

- Notion of ‘optimal’ reductions.
- A general purpose algorithm with optimality guarantees
- Jordan algebra interpretation; hence, easy extension to symmetric cone optimization (e.g., LP, SOCP).
- Combinatorial refinements for computational efficiency
How does symmetry reduction work?

Given SDP \( \min_{X \in \mathcal{A} \cap S^n_+} \text{Tr } CX \), method finds special orthogonal projection \( P : S^n \rightarrow S^n \)

\[ \mathcal{A} \cap S^n_+ \]

If \( X \) feas./optimal, \( P(X) \) feas./optimal.
How does symmetry reduction work?

Given SDP \( \min_{X \in A \cap S^n_+} \text{Tr } CX \), method finds special orthogonal projection \( P : \mathbb{S}^n \to \mathbb{S}^n \)

If \( X \) feasible/optimal, \( P(X) \) feasible/optimal.

\( P \) satisfies following conditions:

\[
P(A) \subseteq A, \quad P(S^n_+) \subseteq S^n_+, \quad P(C) = C
\]
How does symmetry reduction work?

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If \( X \) feas./optimal, \( P(X) \) feas./optimal.

- \( P \) satisfies following conditions:

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P(\mathcal{A}) \subseteq \mathcal{A}, \quad P(\mathbb{S}^n_+) \subseteq \mathbb{S}^n_+, \quad P(C) = C
\]

- Hence, if \( X \) feasible then \( P(X) \) feasible with equal cost:
Example: a MAXCUT SDP relaxation

\[
\begin{align*}
\text{minimize} & \quad \text{Tr } CX \\
\text{subject to} & \quad X \in \mathcal{A} \cap S^n_+ \\
& \quad \mathcal{A} := \{ X \in S^n : X_{ii} = 1 \} \\
& \quad C := \text{adjacency matrix}
\end{align*}
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Let \( \mathcal{G} \) denote group of permutation matrices (automorphisms)

\[ \mathcal{G} := \{ U \text{ a permutation matrix} : U^T CU = C \} \]
Example: a MAXCUT SDP relaxation

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\begin{align*}
\text{minimize} & \quad \text{Tr } C X \\
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Let \( \mathcal{G} \) denote group of permutation matrices (automorphisms)

\[\mathcal{G} := \{ U \text{ a permutation matrix} : U^T C U = C \}\]

Taking \( P(X) := \frac{1}{|\mathcal{G}|} \sum_{U \in \mathcal{G}} U^T X U \), desired conditions hold:

\[P(S_n^+) \subseteq S_n^+ \quad P(\mathcal{A}) \subseteq \mathcal{A}, \quad P(C) = C\]
Example: a MAXCUT SDP relaxation

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Taking \( P(X) := \frac{1}{|\mathcal{G}|} \sum_{U \in \mathcal{G}} U^T X U \), desired conditions hold:

\[
P(\mathbb{S}^n_+) \subseteq \mathbb{S}^n_+ \quad P(\mathcal{A}) \subseteq \mathcal{A}, \quad P(C) = C
\]

Hence, range of \( P \) contains solutions: when \( X \) feasible, \( P(X) \) feasible with equal cost.
Our approach: optimize over projections

Given SDP \( \min_{X \in \mathcal{A} \cap \mathbb{S}_+^n} \langle C, X \rangle \), find map \( P \) that solves

\[
\begin{align*}
\text{minimize} & \quad \text{rank } P \\
\text{subject to} & \quad P(C) = C, \ P(I) = I \\
& \quad P(\mathcal{A}) \subseteq \mathcal{A} \\
& \quad P(\mathbb{S}_+^n) \subseteq \mathbb{S}_+^n \\
& \quad P: \mathbb{S}^n \rightarrow \mathbb{S}^n \text{ an orthogonal projection}.
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\]

Main properties:

- Can be solved in polynomial time (!)
- Range of \( P \) structured: a \textit{Jordan subalgebra} of \( \mathbb{S}^n \).
- \( \mathbb{S}_+^n \cap \text{range } P \) equals a product of symmetric cones.
Invariance characterization of optimal subspace

Theorem (Permenter-P.)

Orthogonal projection $P : \mathbb{S}^n \to \mathbb{S}^n$ solves

\[
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& \quad P(\mathcal{A}) \subseteq \mathcal{A} \\
& \quad P(\mathbb{S}^n_+) \subseteq \mathbb{S}^n_+
\end{align*}
\]

iff the range of $P$ solves

\[
\begin{align*}
\text{minimize} & \quad \dim S \\
\text{subject to} & \quad S \ni \{I, X_{\mathcal{L}^\perp}, C\} \\
& \quad S \ni P_{\mathcal{L}}(S) \\
& \quad S \ni \{X^2 : X \in S\},
\end{align*}
\]

where $\mathcal{A} = X_{\mathcal{L}^\perp} + \mathcal{L}$, and $X_{\mathcal{L}^\perp}$ is the min-norm point of $\mathcal{A}$. 
Subspace optimization and solution algorithm

minimize \( \dim S \)
subject to \( S \ni C, X_L^\perp, I \)
\( S \supseteq P_L(S) \)
\( S \supseteq \{ X^2 : X \in S \} \)

\[ S \leftarrow \text{span}\{ C, X_L^\perp, I \} \]
repeat
\[ S \leftarrow S + P_L(S) \]
\[ S \leftarrow S + \text{span}\{ X^2 : X \in S \} \]
until converged.

Properties of algorithm:
- Optimal subspace contains each iterate (induction)
- Computes ascending chain of subspaces—terminates.
- At termination, subspace feasible; hence, optimal.

Properties of minimization problem:
- Feasible set closed under intersection (lattice)
- A unique solution.
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Properties of minimization problem:
- Feasible set closed under intersection (lattice)
- A unique solution.
Combinatorial descriptions

Great! Now, we can easily compute the optimal subspace $S$. 

- **Partition subspaces**: defined by a partition of $\mathbb{R}^n \times \mathbb{R}^n$.
- **Coordinate subspaces**: defined by a sparsity pattern.
- **Combinatorial subspaces**: orthogonal basis of 0/1 matrices.

E.g.,

\[
\begin{pmatrix}
a & a & b \\
\end{pmatrix}
\begin{pmatrix}
a & a & b \\
\end{pmatrix}
\begin{pmatrix}
b & b & c \\
\end{pmatrix}
\]

vs.

\[
\begin{pmatrix}
a & b & 0 \\
0 & b & c \\
0 & 0 & d \\
\end{pmatrix}
\]

vs.

\[
\begin{pmatrix}
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0 & a & c \\
b & b & b \\
\end{pmatrix}
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Great! Now, we can easily compute the optimal subspace $S$.

But, often want/need additional properties (e.g., “dense” subspaces may not be very efficient).

Can tradeoff dimension with sparsity of a basis?
Combinatorial descriptions

Great! Now, we can easily compute the optimal subspace $S$.

But, often want/need additional properties (e.g., “dense” subspaces may not be very efficient).

Can tradeoff dimension with sparsity of a basis?

Yes! Three kinds of sparse bases for $S$:

- **Partition** subspaces: defined by a partition of $[n] \times [n]$.
- **Coordinate** subspaces: defined by a sparsity pattern
- **Combinatorial** subspaces: orthogonal basis of 0/1 matrices

E.g.,

\[
\begin{bmatrix}
  a & a & b \\
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\end{bmatrix}
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vs.

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  0 & 0 & d \\
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vs.

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Finding optimal structured subspaces

The main algorithm can be adapted to compute the optimal subspace for each of these three cases.
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Key property (again): lattice structure (closedness under intersection)
Finding optimal structured subspaces

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Key property (again): lattice structure (closedness under intersection)

E.g., for partition subspaces, instead of optimizing over lattice of subspaces, use the lattice of partitions:

\[
\begin{align*}
\text{minimize} & \quad \dim S \\
\text{subject to} & \quad S \ni C, X_{\mathcal{L}^\perp}, I \\
& \quad S \ni P_{\mathcal{L}}(S) \\
& \quad S \ni \{X^2 : X \in S\} \\
S & \text{ is a partition subspace}
\end{align*}
\]

\[\mathcal{P} \leftarrow \text{Part}\{C, X_{\mathcal{L}^\perp}, I\}\]

repeat
\[\mathcal{P} \leftarrow \text{refine}(\mathcal{P}, P_{\mathcal{L}})\]
\[\mathcal{P} \leftarrow \text{refine}(\mathcal{P}, X \mapsto X^2)\]

until converged.
Finding optimal structured subspaces

The main algorithm can be adapted to compute the optimal subspace for each of these three cases.

Key property (again): lattice structure (closedness under intersection)

E.g., for partition subspaces, instead of optimizing over lattice of subspaces, use the lattice of partitions:

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& \quad S \ni P_{\mathcal{L}}(S) \\
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& \quad S \text{ is a partition subspace}
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\mathcal{P} \leftarrow \text{Part}\{C, X_{\perp \mathcal{L}}, I\}
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\[\text{repeat}\]

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\end{align*}\]

\[\text{until} \quad \text{converged.}\]

Great! But there’s more...
Decomposition via Jordan algebras

Given SDP $\min_{X \in \mathcal{A} \cap S^n} \langle C, X \rangle$, we’ve found a subspace invariant under $X \mapsto X^2$ containing optimal solutions:

$S \supseteq \{ X^2 : X \in S \}$
Decomposition via Jordan algebras

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\[ S \supseteq \{X^2 : X \in S\} \]

- Subspaces invariant under $X \mapsto X^2$ have decomposition

\[ S = Q \begin{pmatrix} S_1 & 0 & \ldots & 0 \\ 0 & S_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & S_m \end{pmatrix} Q^T, \]

where $S_i$ are simple Jordan algebras.
Decomposition via Jordan algebras

Given SDP \( \min_{X \in \mathcal{A} \cap S^n_+} \langle C, X \rangle \), we’ve found a subspace invariant under \( X \mapsto X^2 \) containing optimal solutions:

Subspaces invariant under \( X \mapsto X^2 \) have decomposition

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\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & S_m
\end{pmatrix} Q^T,
\]

\( S_i \) are simple Jordan algebras

Number of distinct eigenvalues of generic element equals rank of \( S_i \)—a complexity measure.
Minimizing dimension optimizes decomposition

\[
\begin{align*}
\text{minimize} & \quad \dim S \\
\text{subject to} & \quad S \ni X_{L^\perp}, C, I \\
& \quad S \supseteq P_L(S) \\
& \quad S \supseteq \{X^2 : X \in S\},
\end{align*}
\]

All feasible subspaces have decomp. $S = \bigoplus_{i=1}^{d_S} S_i$. In what sense does solution $S^*$ optimize the ranks of each $S_i$?
Minimizing dimension optimizes decomposition

minimize \ dim S
subject to \ S \ni X_{L^\perp}, C, I
S \supseteq P_L(S)
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All feasible subspaces have decomp. \( S = \bigoplus_{i=1}^{d_S} S_i \). In what sense does solution \( S^* \) optimize the ranks of each \( S_i \)?

Thm. (Permenter-P.):
- \( S^* \) minimizes \( \sum_i \text{rank } S_i \) and \( \max_i \text{rank } S_i \)
Minimizing dimension optimizes decomposition

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Thm. (Permenter-P.):
- \( S^* \) minimizes \( \sum_i \text{rank} S_i \) and \( \max_i \text{rank} S_i \)
- \textit{Majorization} inequalities hold, i.e., for each \( m \geq 1 \)
  \[
  \sum_{i=1}^{m} \text{rank} S_i^* \leq \sum_{i=1}^{m} \text{rank} S_i
  \]
  (ranks sorted in decreasing order)
Majorization example

Subspaces (parametrized by $u_i$ and $v_i$) and their rank vectors

$$
\begin{pmatrix}
  u_1 & u_2 & 0 & 0 & 0 \\
  u_2 & u_3 & 0 & 0 & 0 \\
  0 & 0 & u_4 & 0 & 0 \\
  0 & 0 & 0 & u_5 & u_6 \\
  0 & 0 & 0 & u_6 & u_7 \\
\end{pmatrix}
$$

$$
\begin{pmatrix}
  v_1 & v_2 & 0 & 0 & 0 \\
  v_2 & v_3 & 0 & 0 & 0 \\
  0 & 0 & v_4 & v_5 & v_6 \\
  0 & 0 & v_5 & v_7 & v_8 \\
  0 & 0 & v_6 & v_8 & v_9 \\
\end{pmatrix}
$$

$r_u = (2, 1, 2)$

$r_v = (2, 3)$
Majorization example

Subspaces (parametrized by $u_i$ and $v_i$) and their rank vectors

$\begin{pmatrix} u_1 & u_2 & 0 & 0 & 0 \\ u_2 & u_3 & 0 & 0 & 0 \\ 0 & 0 & u_4 & 0 & 0 \\ 0 & 0 & 0 & u_5 & u_6 \\ 0 & 0 & 0 & u_6 & u_7 \end{pmatrix} \quad \begin{pmatrix} v_1 & v_2 & 0 & 0 & 0 \\ v_2 & v_3 & 0 & 0 & 0 \\ 0 & 0 & v_4 & v_5 & v_6 \\ 0 & 0 & v_5 & v_7 & v_8 \\ 0 & 0 & v_6 & v_8 & v_9 \end{pmatrix}$

$r_u = (2, 1, 2) \quad r_v = (2, 3)$

Vector $r'_u = (2, 2, 1)$ majorized by $r'_v = (3, 2, 0)$:

$2 \leq 3, \quad 2 + 2 \leq 3 + 2, \quad 2 + 2 + 1 \leq 3 + 2 + 0$
Jordan algebras

- Jordan algebras are commutative algebras satisfying Jordan identity
  \[(X \circ Y) \circ X^2 = X \circ (Y \circ X^2)\]

- The vector space \(S^n\) a Jordan algebra if equipped with product
  \[X \circ Y := \frac{1}{2}(XY + YX)\]

- The subalgebras of \(S^n\) precisely the sets closed under squaring map \(X \mapsto X^2\) since
  \[XY + YX = (X + Y)^2 - X^2 - Y^2.\]

- Structure theorem of Jordan-von Neumann-Wigner describes subalgebras of \(S^n\)....
Decomposition of $S \cap S^n$

If $S \subset S^n$ is a Jordan subalgebra, it equals direct-sum $\bigoplus_{i=1}^{m} S_i$, where each $S_i$ is isomorphic to one of the following:

- Algebra of Hermitian matrices with real, complex or quaternion entries
- A spin-factor algebra

Implies *cone-of-squares* $S \cap S^n_+$ is isomorphic to product of

- PSD cones with real/complex/quaternion entries
- Lorentz cones

Yields reformulation of original SDP over this product

\[
\begin{align*}
\text{minimize} & \quad \text{Tr } CX \\
\text{subject to} & \quad X \in \mathcal{A} \cap S^n_+ \quad \text{minimize} & \quad \text{Tr } CX \\
& \quad X \in \mathcal{A} \cap \left(T(K_1 \times \cdots \times K_m) \cap S^n_+\right)
\end{align*}
\]
Computational results

Comparison with reduction method of de Klerk ’10 survey (generating *-algebras from data):

<table>
<thead>
<tr>
<th>instance</th>
<th>$S^*$</th>
<th>$S_{data}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>hamming_7_5_6</td>
<td>5</td>
<td>8256</td>
</tr>
<tr>
<td>hamming_8_3_4</td>
<td>5</td>
<td>32896</td>
</tr>
<tr>
<td>hamming_9_5_6</td>
<td>6</td>
<td>131328</td>
</tr>
<tr>
<td>hamming_9_8</td>
<td>6</td>
<td>131328</td>
</tr>
<tr>
<td>hamming_10_2</td>
<td>7</td>
<td>524800</td>
</tr>
</tbody>
</table>

- Table list dimension of our subspace $S^* \subseteq S^n$ and subspace $S_{data} \subseteq S^n$ found by generating *-algebra.
- Decomposing $S^*$ yields a linear program.
## Results: SOSOPT (Seiler ’13) Demo scripts

<table>
<thead>
<tr>
<th>Script Name</th>
<th>$n$ (before)</th>
<th>$n$ (after)</th>
</tr>
</thead>
<tbody>
<tr>
<td>sosoptdemo2</td>
<td>13, 3</td>
<td>$3, 2 \times 3, 1 \times 7$</td>
</tr>
<tr>
<td>sosoptdemo4</td>
<td>35</td>
<td>$5 \times 5, 1 \times 10$</td>
</tr>
<tr>
<td>gsosoptdemo1</td>
<td>9, 5</td>
<td>$6, 3 \times 2, 2$</td>
</tr>
<tr>
<td>IOGainDemo_3</td>
<td>15, 8</td>
<td>$10, 5 \times 2, 3$</td>
</tr>
<tr>
<td>Chesi(1</td>
<td>2)IterationWithVlin</td>
<td>9, 5</td>
</tr>
<tr>
<td>Chesi3_GlobalStability</td>
<td>14, 5</td>
<td>$8, 6, 3, 2$</td>
</tr>
<tr>
<td>Chesi(3</td>
<td>4)IterationWithVlin</td>
<td>9, 5</td>
</tr>
<tr>
<td>Chesi(5</td>
<td>6)_Bootstrap</td>
<td>19, 9</td>
</tr>
<tr>
<td>Chesi(5</td>
<td>6)IterationWithVlin</td>
<td>19, 9</td>
</tr>
<tr>
<td>Coutinho3IterationWithVlin</td>
<td>9, 5</td>
<td>$6, 3 \times 2, 2$</td>
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<tr>
<td>HachichoTibken_Bootstrap</td>
<td>19, 9</td>
<td>$12, 7, 6, 3$</td>
</tr>
<tr>
<td>HachichoTibkenIterationWithVlin</td>
<td>19, 9</td>
<td>$12, 7, 6, 3$</td>
</tr>
<tr>
<td>Hahn_IterationWithVlin</td>
<td>9, 5</td>
<td>$6, 3, 3, 2$</td>
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<tr>
<td>KuChen_IterationWithVlin</td>
<td>19, 9</td>
<td>$13, 6 \times 2, 3$</td>
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<td>Parrilo1_GlobalStabilityWithVec</td>
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<td>$2, 1 \times 3$</td>
</tr>
<tr>
<td>Parrilo2_GlobalStabilityWithMat</td>
<td>3, 2</td>
<td>$2, 1 \times 3$</td>
</tr>
<tr>
<td>VDP_IterationWithVball</td>
<td>5, 4</td>
<td>$3 \times 2, 2, 1$</td>
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<tr>
<td>VDP_IterationWithVlin</td>
<td>9, 5</td>
<td>$6, 3 \times 2, 2$</td>
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<tr>
<td>VDP_LinearizedLyap</td>
<td>9, 5</td>
<td>$6, 3 \times 2, 2$</td>
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<tr>
<td>VannelliVidyasagar2_Bootstrap</td>
<td>19, 9</td>
<td>$13, 6 \times 2, 3$</td>
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<tr>
<td>VannelliVidyasagar2_IterationWithVlin</td>
<td>19, 9</td>
<td>$13, 6 \times 2, 3$</td>
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<tr>
<td>VincentGrantham_IterationWithVlin</td>
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<td>$6, 3 \times 2, 2$</td>
</tr>
<tr>
<td>WTBenchmark_IterationWithVlin</td>
<td>19, 9</td>
<td>$13, 6 \times 2, 3$</td>
</tr>
</tbody>
</table>
Conclusions

New reduction method for SDP.

- Generalizes symmetry reduction and *-algebra-methods
- Fully algorithmic, don’t need to compute automorphisms!
- Yields optimal ‘block-diagonalization’ (majorization)
- Can exploit combinatorial description of subspace
- Through Jordan algebra theory, extends to LP/SOCP/...
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Preprint at \texttt{arXiv:1608.02090}.

Thanks for your attention!