Geometric Reasoning in 3D Environments Using SOS Programming

Vikas Sindhwani
Google Brain
sindhwani@google.com

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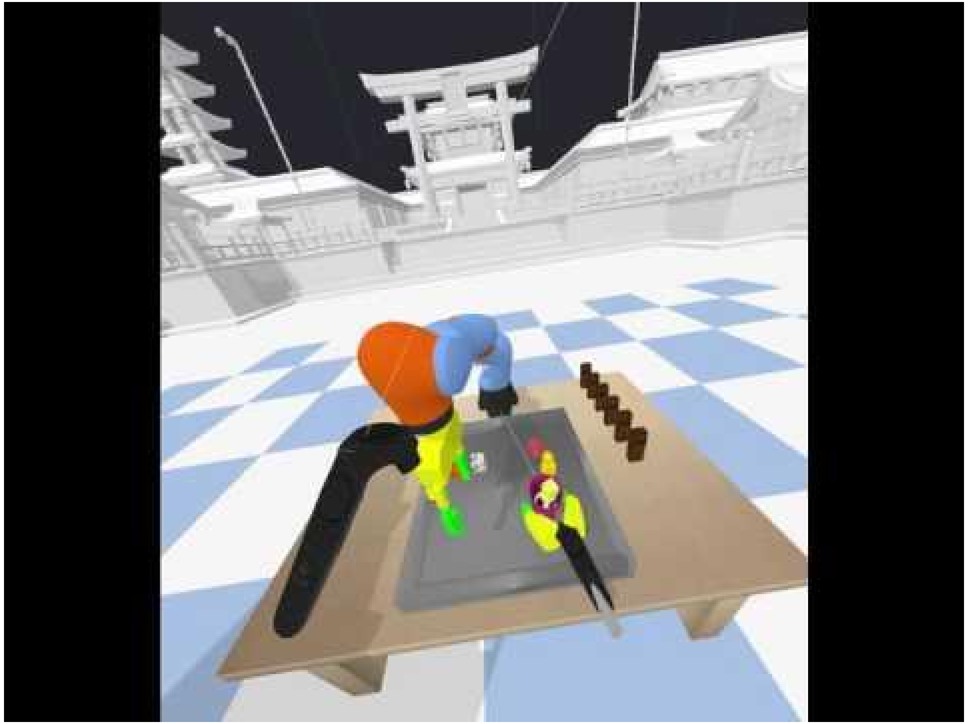
[Geometry of 3D Environments and Sum of Squares Polynomials (https://arxiv.org/pdf/1611.07369v1.pdf)](https://arxiv.org/pdf/1611.07369v1.pdf)

Perception → Geometry → Control

Manipulation: Making and Breaking Contact Optimally

Virtual Reality: Blending Reality and Simulation

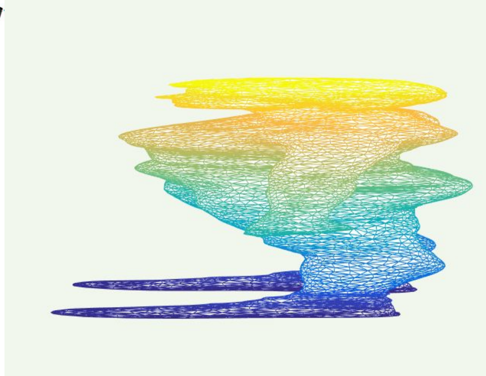


By Erwin Coumans (Google Brain)
Creator of [Bullet Physics Engine]

High Level Perspectives

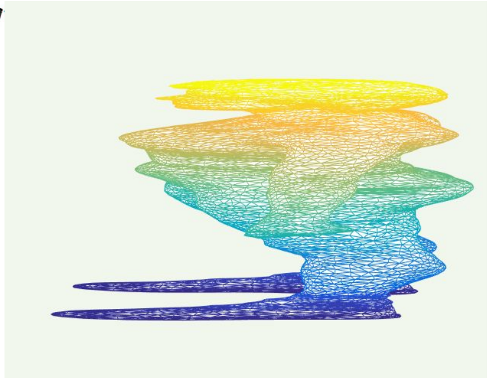
- Understanding Geometric Proximity relationships between the robot and the environment in real-time.
  - Search → Perception → Embodiment
    - Human-machine physical contact is a profound paradigm shift
  - Safety Guarantees are very important - huge difference from search.
- Key Concepts from Sum-of-Squares Optimization
  - Search for convex and near-convex polynomials whose sublevel sets tightly contain 3D regions.
    - SOS-Convexity and generalizations of the Lowner-John Minimum ellipsoid problem.
- Focus on small-scale but potentially real-time Semidefinite Programming
  - How practical is SOS programming, given its scalability challenges, in this context, where n=3?

Perception → Geometry → Control

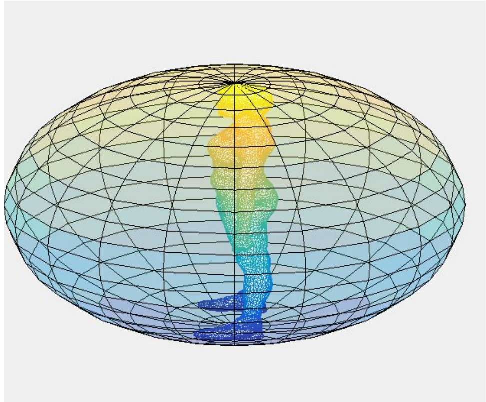


$$D = \{x_1, x_2, \dots x_m\}$$

Perception → Geometry → Control



Safety Shield



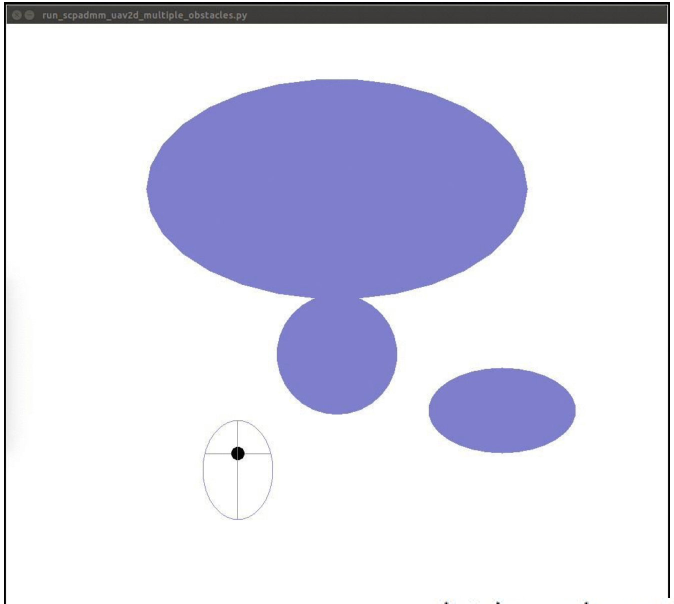
$$D = \{x_1, x_2, \ldots x_m\} \quad \subseteq \quad S_p = \{x \in \mathbb{R}^3 | p(x) \leq 1\}$$

Perception → Geometry → **Control**

$$\min_{u_0,\ldots u_T} \sum_{t=0}^{T} c_t(x_t, u_t) + c_T(x_T)$$

subject to:

$$x_{t+1} = f(x_t, u_t)$$



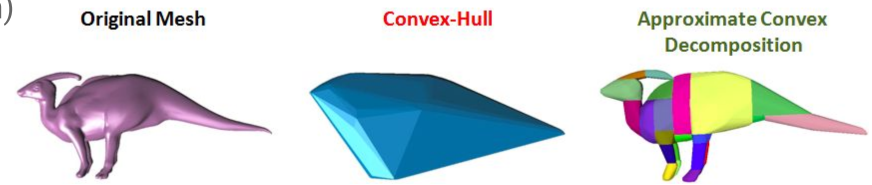
$$\frac{d\mathbf{x}}{dt} = f(\mathbf{x}, \mathbf{u}) = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} -v\ \sin(\theta) \\ v\ \cos(\theta) \\ u \end{pmatrix}$$

$$\text{distance}(S^i_{p_\text{robot}}(x_t), S^j_{p_\text{env}}) >= \text{safety-margin}$$

- Nested optimization, possibly MPC style to handle dynamic environment
- Convexity of bounding volume
- Need generalized notions of distance that allow penetration

A Collection of Geometry Problems for Robot Control

- Depth Camera → 3D Point Cloud → Bounding Volume (BV)
  - Describe Robot body & Environment ("mid-level" vision)
  - Tight Containment and Minimum Volume
  - Fast Construction Time (e.g., new objects appear)
  - Fast Reconstruction upon Rigid body motion
- Distance and Collision Queries for Path Planning
  - Point to BV (e.g., Is a specific voxel safe?)
  - Distance between bounding volumes for Trajectory Optimization
  - Handle overlaps, i.e., Measure of Penetration, e.g., "penetration depth"
- Handling Non-convexity
  - Convex Decomposition of Objects
  - Tradeoff level of convexity of BV with tightness
- Outer-approximate a set of BVs with a single convex BV.
  - Can be used to define a BV Hierarchy (coarse-to-fine representation)
  - Convexification of a nonconvex body
- Several others, e.g. Chebyschev Centers (e.g., safest points, in conjunction with convex decomposition of free space)



From: https://github.com/kmammou/v-hacd

Geometry of 3D Environments and SOS Programming

- Define Bounding Volumes using sublevel sets of SOS polynomials.
  - SOS formulation gives effectives heuristics for minimizing volume
  - Impose Convexity on sublevel sets via SOS-Convexity
- Nonnegative Polynomials and SOS sufficient condition.

$$p(x) = \sum_{\alpha} c_\alpha x^\alpha \geq 0 \; \forall \; x \in \mathbb{R}^n \Longleftarrow p(x) = \sum_{i=1}^{k} q_i^2(x) = z(x)^T Q z(x), Q \succeq 0$$

- Convex Polynomials and SOS-Convex sufficient condition.

$$\nabla^2 p(x) \succeq 0 \; \forall \; x \in \mathbb{R}^n \Longleftarrow y^T \nabla^2 p(x) y = z(x,y)^T Q z(x,y), Q \succeq 0$$

  - Complete characterization of the Gap (Ahmadi and Parrilo, 2012): “the remarkable outcome is that convex polynomials are sos-convex exactly in cases where nonnegative polynomials are sums of squares…”
- SOS Bodies & SOS-Convex Bodies:

$$S_p = \{x \in \mathbb{R}^3 | p(x) \leq 1\} \quad p \text{ sos, or, sos-convex}$$

Minimum Volume SOS-Convex Bodies

Generalization of Minimum Volume Ellipsoids (Lowner-John Problem)

- Maximum Curvature Formulation
  - Magnani, Lall and Boyd, 2005

$$\min_{p \in \mathbb{R}_{2d}[x], H \in S^{\tilde{N} \times \tilde{N}}} -\log \ \det(H)$$

subject to:

$$p = z(x)^T P z(x), \ \ P \succeq 0$$

$$y^T \nabla^2 p(x) y = w(x,y)^T H w(x,y), \ \ H \succeq 0$$

$$p(x_i) \leq 1, i = 1, \ldots, m$$

- Our Formulation

$$\min_{p \in \mathbb{R}_{2d}[x], P \in S^{N \times N}} -\log \ \det(P)$$

subject to:

$$p(x) = z(x)^T P z(x), \ \ P \succeq 0$$

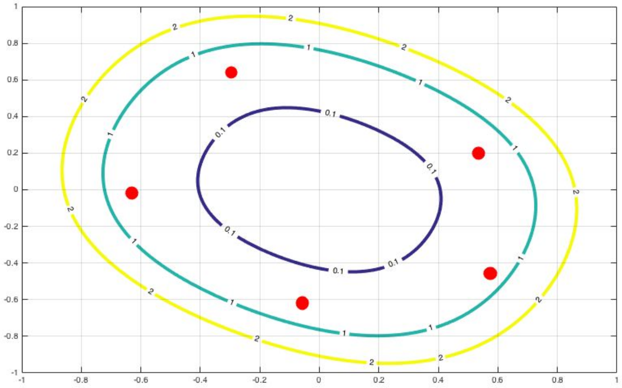
$$y^T \nabla^2 p(x) y = w(x,y)^T H w(x,y), \ \ H \succeq 0$$

$$p(x_i) \leq 1, i = 1, \ldots, m$$

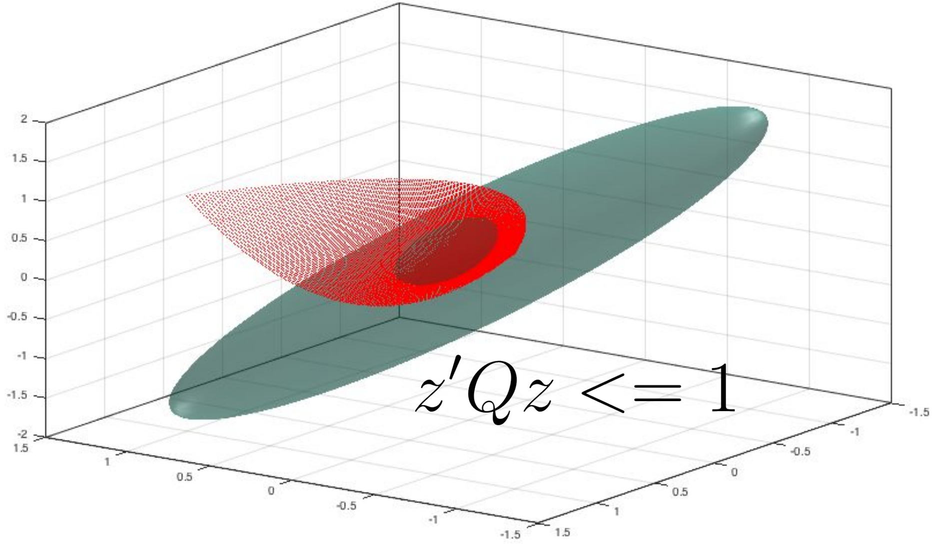
- Both exact for quadratic case (2d=2).
- Both heuristic for higher (2d>=4), and note curvature might be maximized in directions of no data.

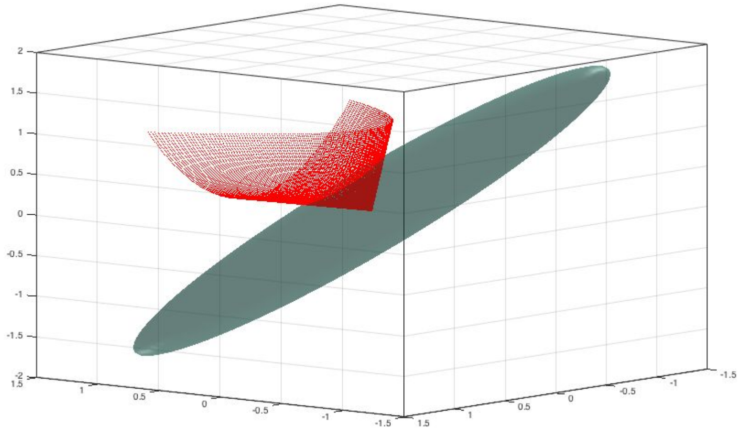
Justification

$$p(x, y) = z(x, y)^T Q z(x, y), \quad Q \succeq 0$$



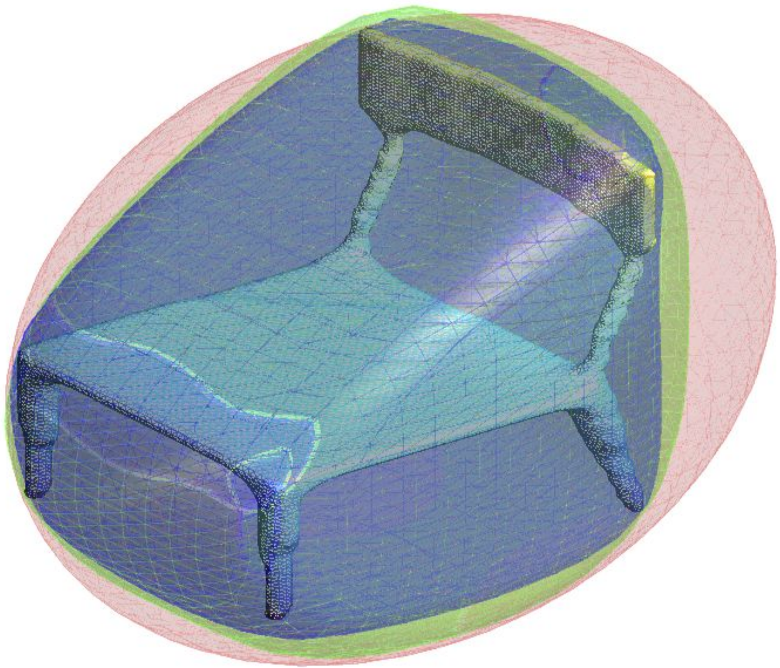
$$z(x, y) = \begin{pmatrix} x^2 \\ xy \\ y^2 \end{pmatrix}$$



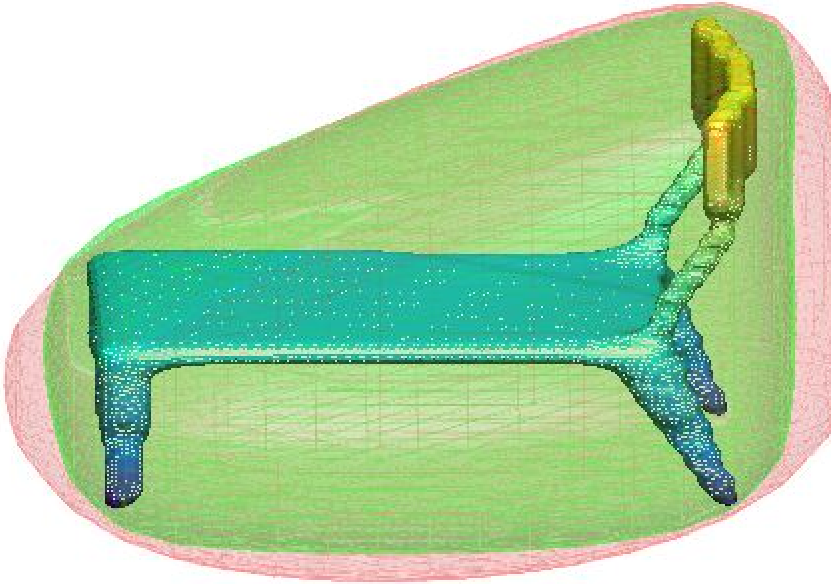


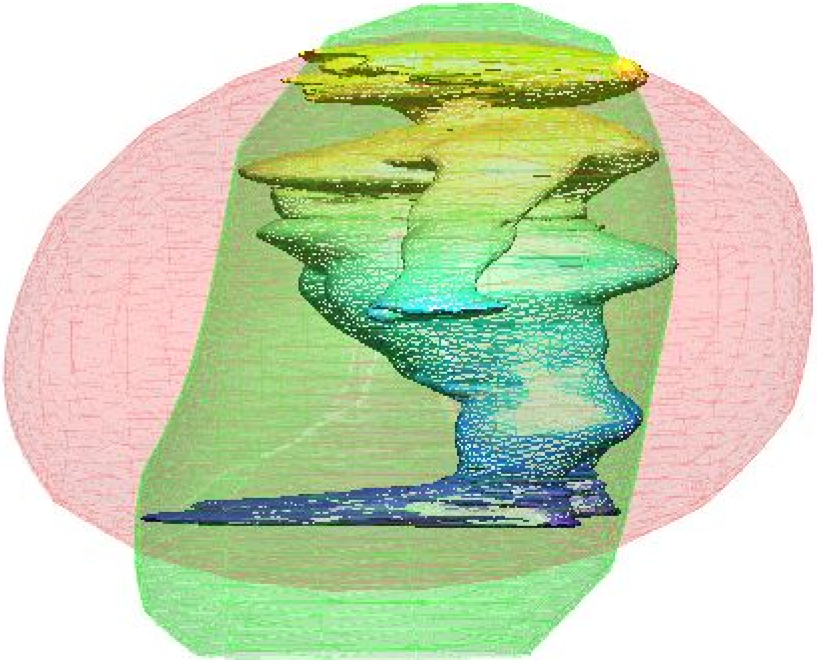
An SoS-Convex Chair

Degrees 2, 4, 6



Visual Comparison with Maximum Curvature Heuristic





Relaxing Convexity

- Inverse Moment Matrix Formulation
  - Lasserre and Pauwels, 2015

$$p(x) := z(x)^T M_d z(x)$$

$$M = \left(\frac{1}{m} \sum_{i=1}^{m} z(x_i) z(x_i)^T \right)^{-1}$$

  - Very fast and effective single pass method
  - Minimizes average value on the point cloud, but not the volume explicitly.
  - Sublevel-set value needs to be tuned for point cloud containment.
  - No direct control over level of convexity.

- Our Formulation

$$\min_{p \in \mathbb{R}_{2d}[x], P \in S^{N \times N}} -\log\ \det(P)$$

$$\text{subject to:}$$

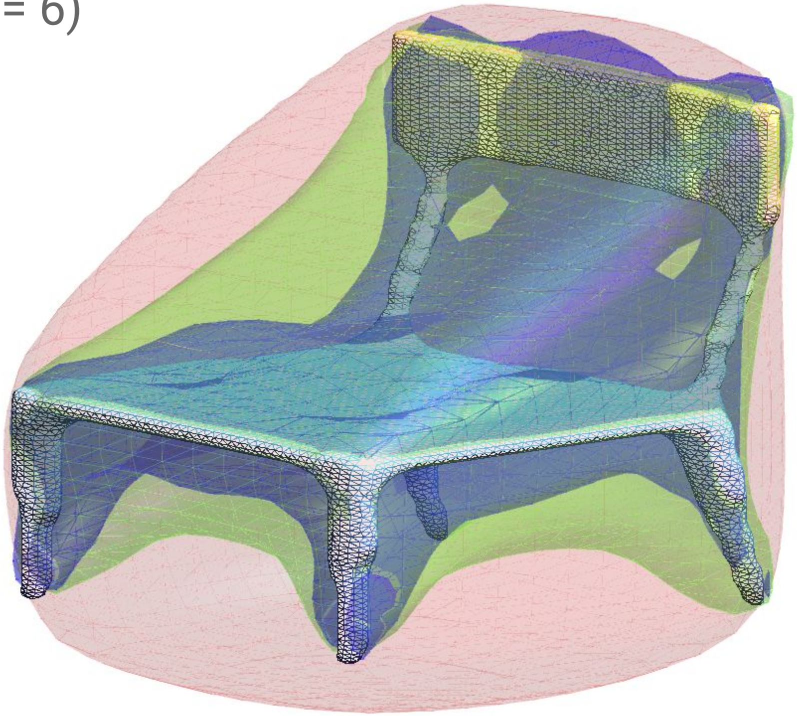
$$p(x) = z(x)^T P z(x),\ \ P \succeq 0$$

$$p(x) - c(\sum_i x_i^2)^d \ \ \text{sos-convex}$$

$$p(x_i) \leq 1, i = 1, \ldots, m$$

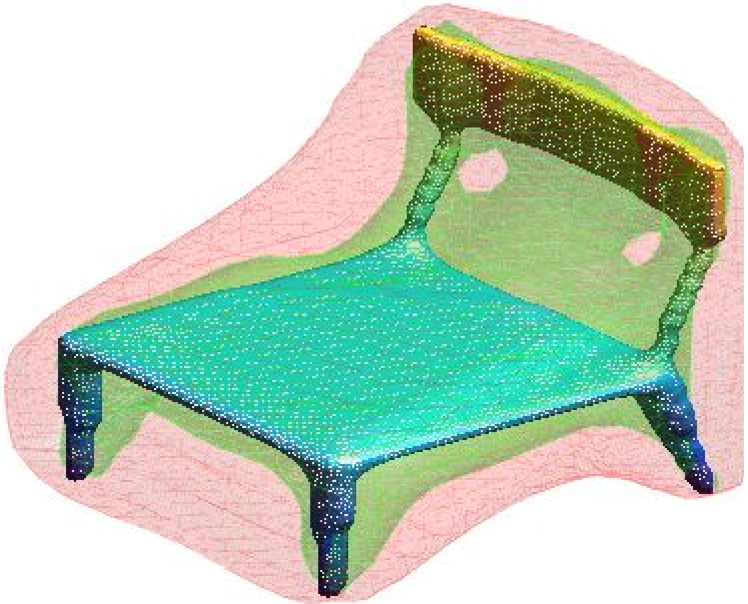
SoS Chairs with relaxed convexity

c=0, -10, -100 (degree = 6)



Visual Comparison with Inverse Moment Approach

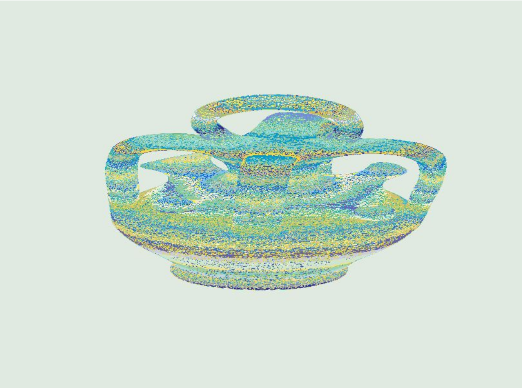
degree 6, c=-100

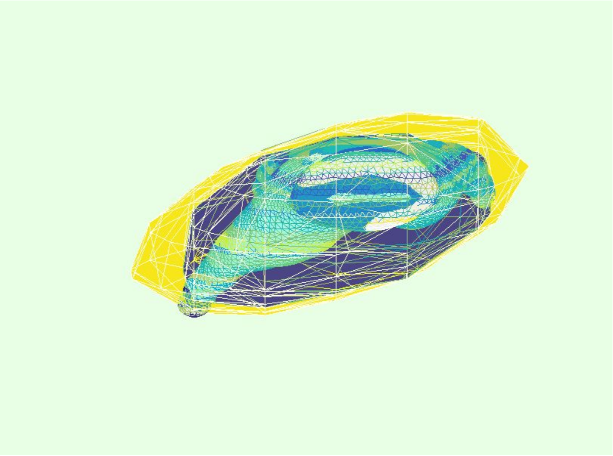


Bounding Volume Effectiveness

	Object → Bounding Body ↓	id:#vertices	Human 10:9508	Chair 101:8499	Hand 181:7242	Vase 361:14859	Octopus 121:5944
CONVEX	Convex-Hull		0.29 (364)	0.66 (320)	0.36 (652)	0.91 (1443)	0.5 (414)
	Sphere		3.74	3.73	3.84	3.91	4.1
	AABB		0.59	1.0	0.81	1.73	1.28
	sos-convex (2)	*logdet*	0.58	1.79	0.82	1.16	1.30
		trace	0.97	1.80	1.40	1.2	1.76
	sos-convex (4)	$logdet(\mathbf{H}^{-1})$	0.57	1.55	0.69	1.13	1.04
		$trace(\mathbf{H}^{-1})$	0.56	2.16	1.28	1.09	3.13
		$logdet(\mathbf{P}^{-1})$	0.44	1.19	0.53	1.05	0.86
		$trace(\mathbf{P}^{-1})$	0.57	1.25	0.92	1.09	1.02
	sos-convex (6)	$logdet(\mathbf{H}^{-1})$	0.57	1.27	0.58	1.09	0.93
		$trace(\mathbf{H}^{-1})$	0.56	1.30	0.57	1.09	0.87
		$logdet(\mathbf{P}^{-1})$	0.41	1.02	0.45	0.99	0.74
		$trace(\mathbf{P}^{-1})$	0.45	1.21	0.48	1.03	0.79
NON-CONVEX	Inverse-Moment (2)		4.02	1.42	2.14	1.36	1.74
	Inverse-Moment (4)		1.53	0.95	0.90	1.25	0.75
	Inverse-Moment (6)		0.48	0.54	0.58	1.10	0.57
	sos (d=4, c=-10)	$logdet(\mathbf{P}^{-1})$	0.38	0.72	0.42	1.05	0.63
		$trace(\mathbf{P}^{-1})$	0.51	0.78	0.48	1.11	0.71
	sos (d=6, c=-10)	$logdet(\mathbf{P}^{-1})$	0.35	0.49	0.34	0.92	0.41
		$trace(\mathbf{P}^{-1})$	0.37	0.56	0.39	0.99	0.54
	sos (d=4, c=-100)	$logdet(\mathbf{P}^{-1})$	0.36	0.64	0.39	1.05	0.46
		$trace(\mathbf{P}^{-1})$	0.42	0.74	0.46	1.10	0.54
	sos (d=6, c=-100)	$logdet(\mathbf{P}^{-1})$	0.21	0.21	0.26	0.82	0.28
		$trace(\mathbf{P}^{-1})$	0.22	0.30	0.29	0.85	0.37

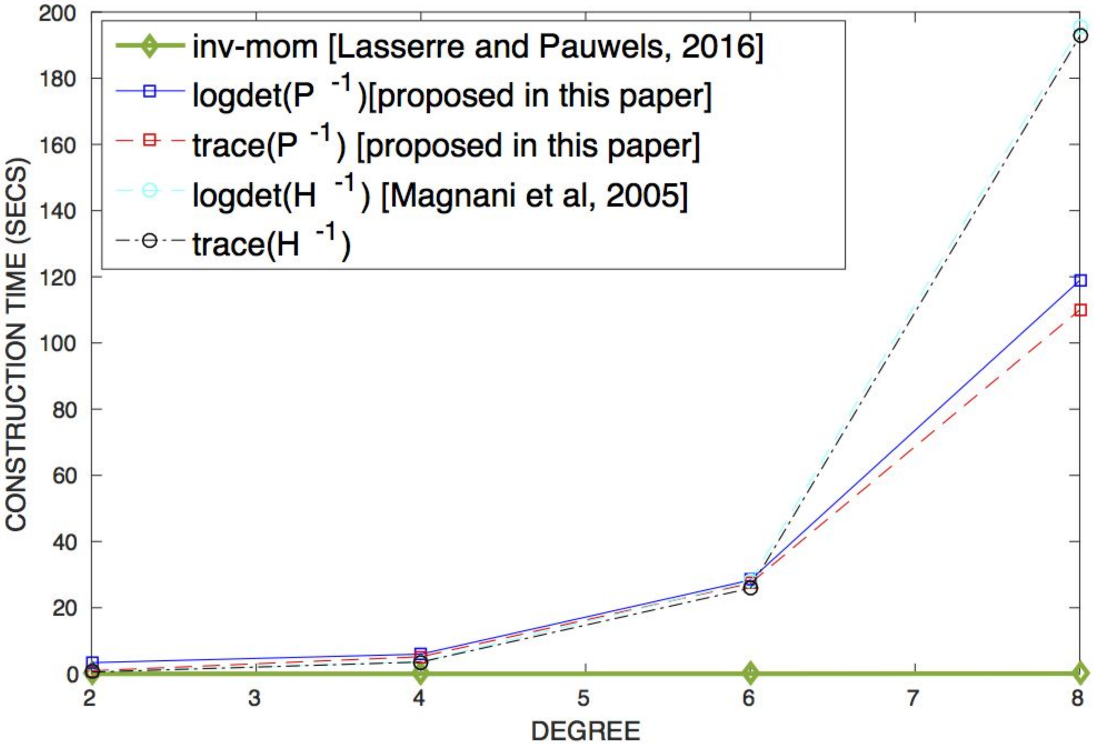
TABLE I: Comparison of various bounding volume techniques





Construction Time

- YALMIP + SCS/ADMM
  - 2500 iterations



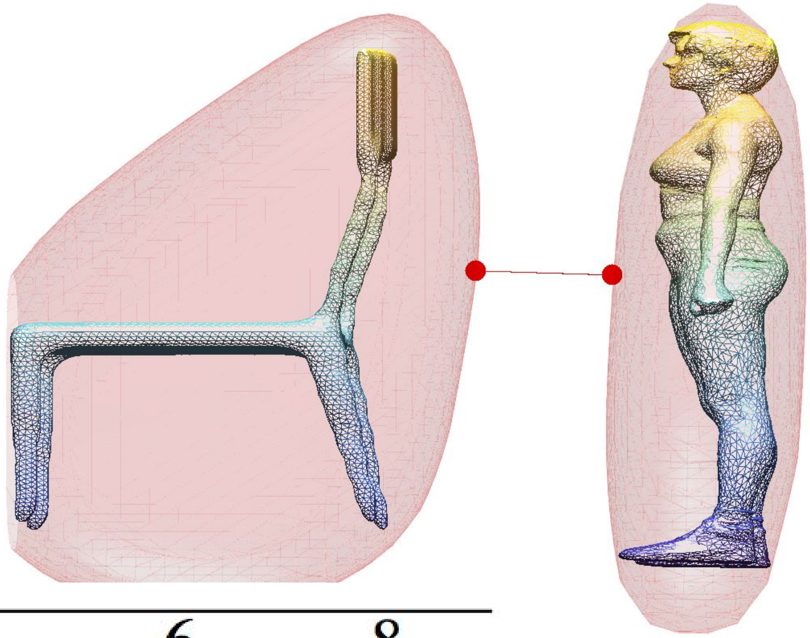
- Note: One-time SOS solution -- if 3D body represented by p(x)<=1, rotates, translates by (R, t), then p(R'x - R't) is the new representation.

Euclidean Distance between SOS-Convex Bodies

- Distance Computation via Convex Optimization

$$\min_{x \in S_{p_1}, y \in S_{p_2}} \|x - y\|_2^2$$

- Near real-time performance with a general-purpose interior-point convex optimizer.
  - Many optimizations possible.

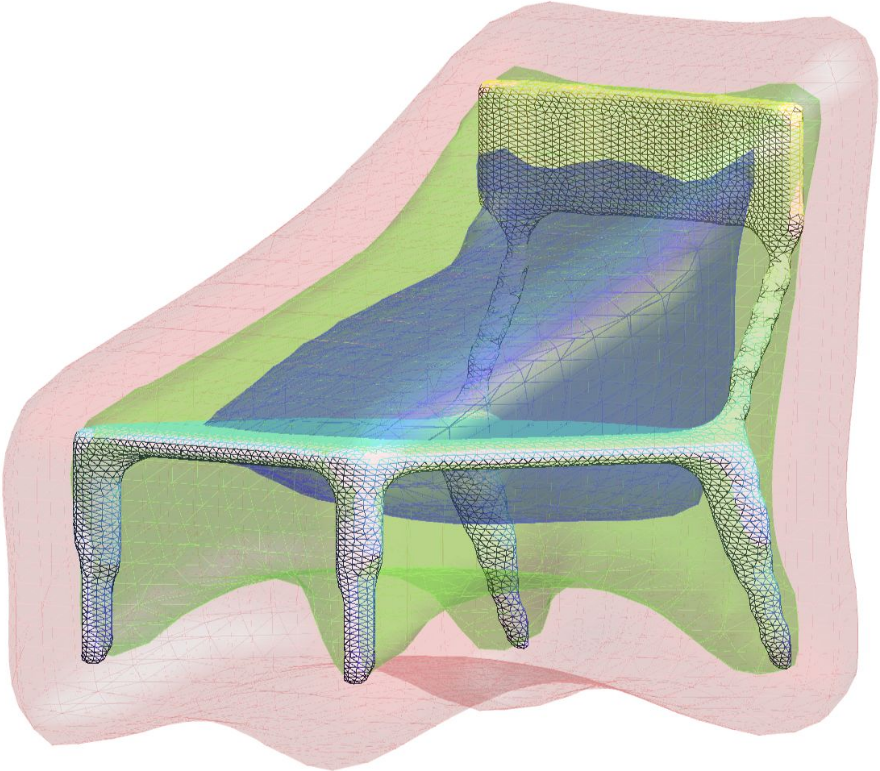


degree	2	4	6	8
time (secs)	0.08	0.083	0.13	0.34

Growing and Shrinking SOS Chair

Level sets: 2, 1, 0.75

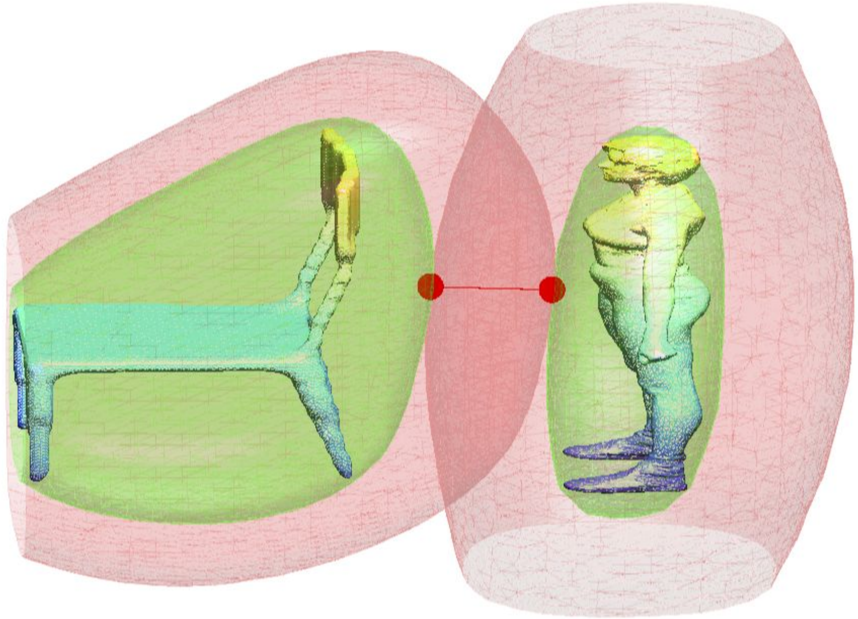
(degree 6, c=-10)

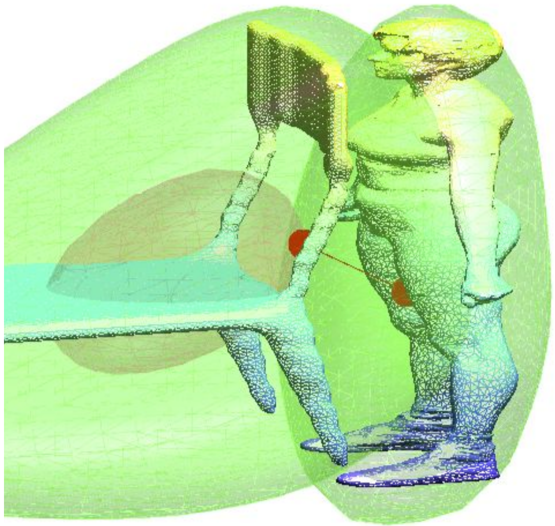


New Measures of Separation and Penetration

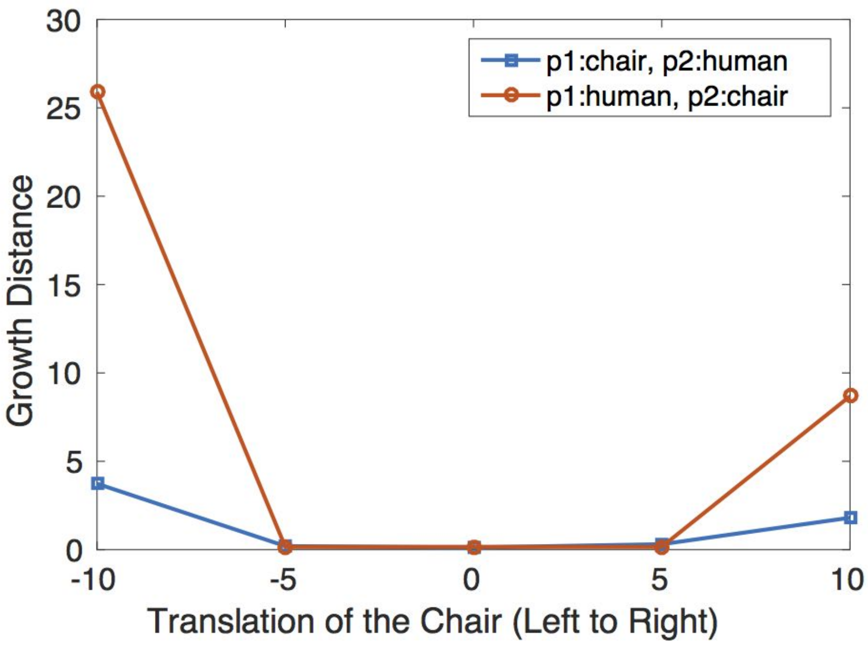
$$d(p_1||p_2) = \min p_1(x)$$
$$\text{s.t. } p_2(x) \leq 1.$$

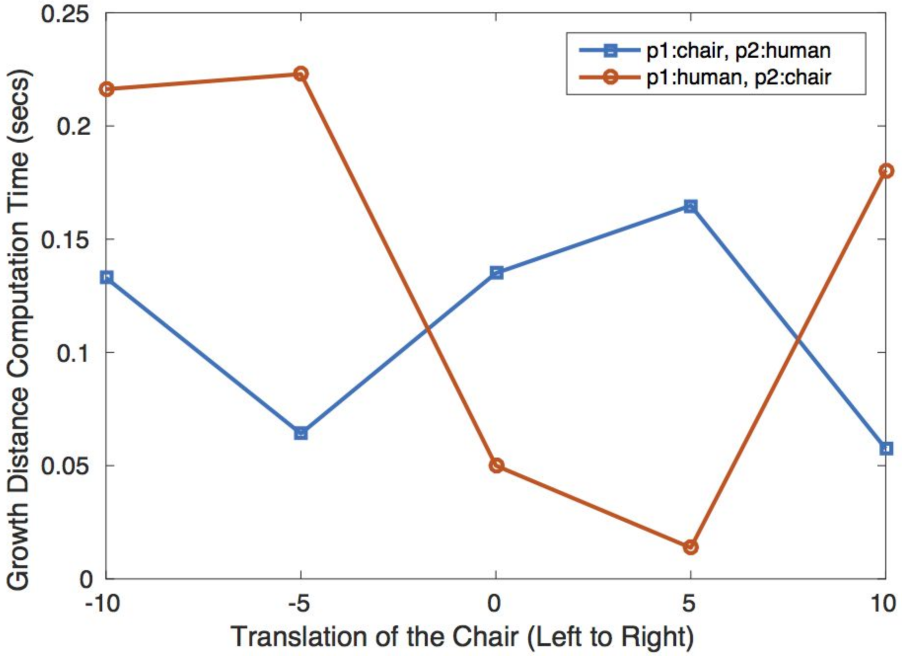
- if $d(p_1||p_2) > 1$, the bounding volumes are separated.
- if $d(p_1||p_2) = 1$, the bounding volumes touch.
- if $d(p_1||p_2) < 1$, the bounding volumes overlap.





Real-time Performance





Containment of Polynomial Sublevel Sets

- Convexification
- BVH: Coarser representations

$$\min_{p \in \mathbb{R}_{2d}[x], \tau_i \in \mathbb{R}_{2\hat{d}}[x], P \in S^{N \times N}} \; -\log\det(P)$$

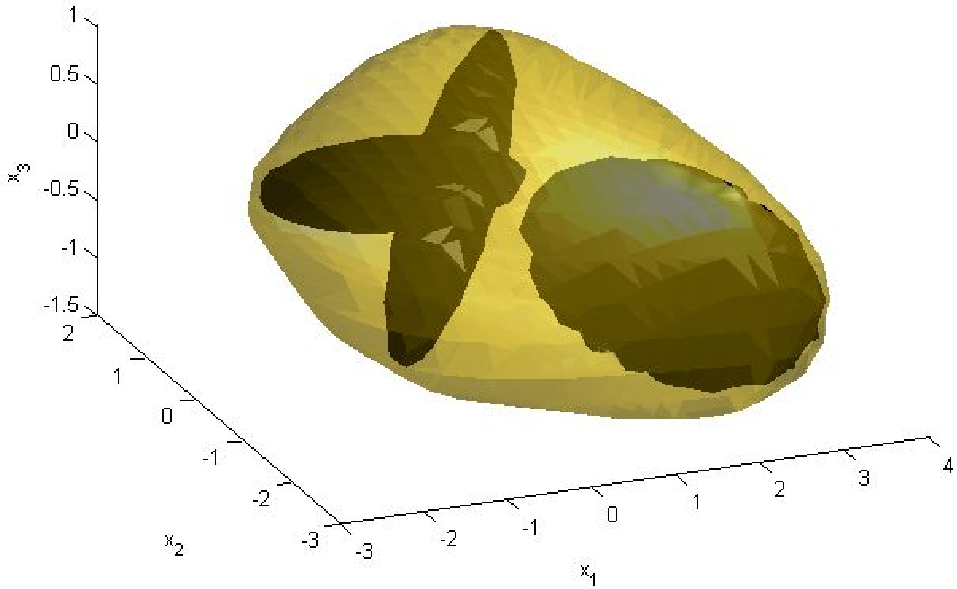
s.t.

$$p(x) = z(x)^T P z(x), P \succeq 0,$$

$$p(x) \quad \text{sos-convex},$$

$$1 - p(x) - \sum_{i=1}^{m} \tau_i(x)(1 - g_i(x)) \quad \text{sos},$$

$$\tau_i(x) \quad \text{sos}, \quad i = 1, \ldots, m.$$



Summary

- Sum of Squares Optimization is practical for an important class of 3D Geometry Problems in Robotics
    - Introduced a new effective bounding volume technique based on SOS-Convexity
    - Small SDPs in this context can be solved fast - stable upto degree 8.
- Constructing 3D representations from real streaming RGBD datasets
- Study interplay between geometry and optimization
    - Integrating such representations with Optimal Control