Objectives of the homework: The goal of this homework is to give you some practice with modeling and basic use of MATLAB, as well as refresh your memory on some concepts in multivariate calculus and linear algebra that are going to come up in the course.

Problem 1: A “huuuge” optimization problem for Donald Trump

You have been offered to serve as a junior campaign manager for Donald Trump, a position you may or may not want to accept, but kind of have to for the purposes of this problem set. You’ve been given access to a total budget of $450,000, which you need to allocate judiciously. Your goal is to maximize the number of voters attracted, and your options for spending the money are the following:

- You could buy Mr. Trump private advising sessions on Foreign Policy. Each hour of private advising will bring in 2,300 new voters\(^1\) up to 150 hours, but will cost $3,000. You can also schedule advising for any fraction of the hour; the cost and the number of voters will be proportional (ignore round-off issues).

- You could hire a tanning expert and a hairdresser specializing in toupees for the entire length of the campaign. This would be a one-time cost of $100,000 and the results are expected to bring in approximately 120,000 voters.

- You could decide to run a campaign video on either CNN or Fox News, or neither. Broadcasting on Fox News would set you back by $250,000 but would bring in 230,000 voters, whereas the cost to broadcast on CNN would be $200,000, with 190,000 voters. Due to exclusivity contracts, you cannot broadcast on both channels.

Write down the optimization problem you are facing by using only linear and/or quadratic functions. Clearly specify your decision variables, constraints, and objective function. (You don’t need to solve the problem.)

\(^1\)It is the job of a statistician to extract these numbers from past observations, or more generally to go from data to models. The job of an optimizer is to go from models to decisions. (Quotes stolen from Prof. William Massey.) In recent years though, it is increasingly common for statisticians to also use optimization to improve the accuracy of their models.
Problem 2: Gradients and Hessians

1. Let \( f(x_1, x_2) = x_1^4 + x_2^4 - x_1 x_2^3 - x_2^2 x_1^3 + x_1^3 + x_2^3. \)

   (a) Compute the gradient \( \nabla f \) and the Hessian \( \nabla^2 f \) of \( f \) by hand. Is the Hessian positive semidefinite, positive definite, negative definite, negative semidefinite, or indefinite at the following points: \((1,0)^T\) and \((1,1)^T\)? Justify your answer without using MATLAB.

   (b) Plot the function in MATLAB using the `ezsurf` function. Recall that to plot \( f(x, y) = x^2 - y^2 \), the MATLAB code can be the following:

   ```matlab
   syms x y; % declare variables
   ezsurf(x^2-y^2);
   ```

   (c) Define level sets. Plot the level sets of the function \( f \) above. Add its gradient vectors to your graph. What can you say about the orientation of the gradient vectors with respect to the level sets? We recommend you use the following meshgrid in MATLAB for plotting:

   ```matlab
   x = -5:0.2:5;
y = -5:0.2:5;
[x1,x2]=meshgrid(x,y);
   ```

   (The meshgrid fixes a grid on which the function gets evaluated.) Additionally, you may want to use the functions `gradient`, `contour`, `eval` and `quiver` in MATLAB. See the documentation for examples on how to use these functions by typing in `doc contour` or `help contour`. The MATLAB code `hold on` enables you to plot multiple figures one one graph.

2. Let \( f(x_1, x_2, x_3, x_4) = x_1^4 \cdot x_2 - x_3 \cdot \frac{1}{(1+x_2)^2} + 100 \cdot x_1 \cdot e^{x_3} + x_4^3. \)

   (a) Compute the gradient of \( f \) and the Hessian of \( f \) using MATLAB.

   (b) Is the Hessian positive semidefinite, positive definite, negative definite, negative semidefinite, or indefinite at the following points: \((1,1,-5,0)^T\), \((1,1,-5,2)^T\) and \((1,1,1,2)^T\)? Justify numerically by using the `eig` function in Matlab that returns the eigenvalues of a matrix.

Problem 3: Taylor expansions

1. Consider the function \( f(x) = \sin(x) \).
(a) Compute the derivatives of \( f \) up to the third order and write the 0\(^{th} \), 1\(^{st} \), 2\(^{nd} \) and 3\(^{rd} \) Taylor approximation of \( f \) around \( x = 0 \).

(b) Plot \( f \) as well as its Taylor approximations computed above on the same graph but with different colors. You can use \texttt{plot}, \texttt{ezplot}, or \texttt{fplot} to do this.

(c) Plot \( \frac{|f - f_3|}{|x|} \), where \( f_3 \) denotes the 3\(^{rd} \) Taylor approximation of \( f \) around \( x = 0 \). Does this agree with what Taylor’s theorem tells you? (See pages 74-75 of the CZ book or Lecture 2 in the notes.) Explain.

2. Let \( f(x_1,x_2) = (x_1 + x_2)^2 - (x_1 - x_2)^5 \). Derive the 1\(^{st} \) order and 2\(^{nd} \) order Taylor approximations of \( f \) around point \((2,1)\). Plot \( f \) and the hyperplane corresponding to its first order Taylor approximation on to the same graph; then plot \( f \) and the quadratic function corresponding to the second order Taylor approximation on to the same graph.

**Problem 4: Unit spheres of different norms**

The unit sphere of a norm \( f : \mathbb{R}^n \to \mathbb{R} \) is defined as its 1-level set, i.e., \( \{ x \in \mathbb{R}^n \mid f(x) = 1 \} \). Draw the unit sphere of the 1-norm, the 2-norm, and the \( \infty \)-norm on the plane (i.e., when \( n = 2 \)) by hand. In other words, you have to plot the following sets:

\[
S_1 = \{ x = (x_1, x_2) \in \mathbb{R}^2 \mid ||x||_1 = 1 \} \\
S_2 = \{ x = (x_1, x_2) \in \mathbb{R}^2 \mid ||x||_2 = 1 \} \\
S_\infty = \{ x = (x_1, x_2) \in \mathbb{R}^2 \mid ||x||_\infty = 1 \}.
\]

Show your work.

**Problem 5: Perfect numbers, roses, and practice with for loops**

A positive integer is said to be perfect if it equals the sum of its divisors, excluding itself. For example, the number 6 is perfect because these divisors for 6 are 1, 2, 3 and we have \( 1 + 2 + 3 = 6 \). The number 10 is not perfect since \( 1 + 2 + 5 = 8 \neq 10 \).

1. You have been thinking about buying red roses for the person you love. You want the number of roses that you give to be perfect. This would convey the romantic message that (s)he is perfect in your eyes. Suppose you are willing to buy up to 1000 roses (yes, love can make you do strange things), write a MATLAB code that tells you all your options. (To get credit, your code should produce the result. You may want to use the MATLAB function \texttt{mod}.)

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2. In many cultures around the world, it is customary to give flowers in odd numbers. In Russia, e.g., an even number of flowers is usually only given at funerals. In the Persian culture, one gives an odd number of flowers to let the receiver know that there will be a future time with another odd number of flowers to “even up” the flowers he/she just received. Suppose you want to give your loved one a number of red roses which is both odd and perfect. What would be the smallest such number? (It is OK to give the best lower bound achieved after 10 minutes on your machine.)