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PRINCETON UNIVERSITY

ORF 363/COS 323
Final Exam, Fall 2015

JANUARY 13, 2016

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AIs: G. Hall, H. Hao, J. Ye, Z. Zhu

1. Please write out and sign the following pledge on top of the first page of your exam: "I pledge my honor that I have not violated the Honor Code or the rules specified by the instructor during this examination."
2. Don't forget to write your name on the exam. Make a copy of your solutions and keep it.
3. The exam is not to be discussed with *anyone* except possibly the professor and the TAs. You can only ask *clarification questions*, and only as *public* (and preferably non-anonymous) questions on Piazza. No emails.
4. You are allowed to consult the lecture notes, your own notes, the reference books of the course as indicated on the syllabus, the problem sets and their solutions (yours and ours), the midterm and its solutions (yours and ours), the practice midterm and final exams and their solutions, all Piazza posts, but *nothing else*. You can only use the Internet in case you run into problems related to MATLAB or CVX. (There should be no need for that either.)
5. You are allowed to refer to facts proven in the notes or problem sets without reproving them.
6. For all problems involving MATLAB or CVX, show your code. The MATLAB output that you present should come from your code.
7. Unless you have been granted an extension because of overlapping finals, the exam is to be turned in on Friday (January 15, 2016) at 9 AM in the instructor's office (Sherrerd 329). If you cannot make it on Friday and decide to turn in your exam sooner, you have to drop it off in the ORF 363 box of the ORFE undergraduate lounge (Sherrerd 123). If you do that, you need to *write down the date and time on the first page of your exam and sign it.* You can also submit the exam electronically on Blackboard. ——— Good luck!

Grading

Problem 1	20 *pts*	
Problem 2	20 *pts*	
Problem 3	20 *pts*	
Problem 4	20 *pts*	
Problem 5	20 *pts*	
TOTAL	100	

Problem 1: To sleep or not to sleep? A student is taking a 48-hour take-home exam and would like to understand how his/her grade will be affected by the number of hours he/she devotes to sleep. Based on past surveys, the instructor has provided the class with a set of data points (t_i, y_i), where y_i is the average grade (out of 100) of previous students who slept t_i hours during the course of the exam. These data points, for $i = 1, \ldots, 7$, are as follows:

$$\{(0, 67), (8, 88), (14, 90), (24, 85), (32, 68), (40, 57), (48, 0)\}.$$

(a) Statisticians have shown that the relationship between y and t is well described by a quadratic function. Your job is to fit a quadratic polynomial

$$p(t) = c_2 t^2 + c_1 t + c_0$$

to the data given above that solves the following optimization problem:

$$\underset{c_0, c_1, c_2}{\text{minimize}} \quad \sum_{i=1}^{7} (p(t_i) - y_i)^2. \tag{1}$$

This is a least squares problem. It can be written as the minimization of a quadratic function

$$\frac{1}{2} c^T Q c - b^T c + r,$$

where Q is a 3×3 matrix, $b \in \mathbb{R}^3$, $r \in \mathbb{R}$, and $c = (c_0, c_1, c_2)^T$. Solve problem (1) by applying the conjugate gradient algorithm. Start the algorithm at the point $(0, 0, 0)^T$ and stop the iterations when the 2-norm of the gradient of the objective function becomes smaller than 10^{-5}. Report the optimal value and optimal solution of (1) and plot the optimal quadratic polynomial along with the data points in the range $t \in [0, 48]$. How many iterations did your algorithm run for? What is the condition number of Q?

(b) Unless there is some bribery going on, a student who sleeps 48 hours during the exam deserves nothing but a zero. There is no guarantee that the least squares fit will respect this feature even though our last data point is $(48, 0)$. Use CVX to resolve problem (1) subject to the following constraint:

$$p(48) = 0. \tag{2}$$

Report the new optimal value, and plot the new optimal quadratic polynomial along with the data points in the range $t \in [0, 48]$. Based on the curve you obtain, what is optimal number of hours for the student to sleep (to one significant digit)?

Problem 2: Consider the set

$$S = \{(x_1, x_2) \in \mathbb{R}^2 |\ (x_2 \geq 0 \text{ and } x_2 \leq x_1 - 1) \text{ or } (x_2 \leq 0 \text{ and } x_2 \geq x_1 - 1)\}.$$

(a) Is S convex? Why or why not?

(b) Let $x = (x_1, x_2)^T$. Find a matrix A, a vector b, and a scalar c such that

$$S = \{x |\ x^T Ax + b^T x + c \leq 0\}.$$

Can S be written in this form with a positive semidefinite matrix A? Why or why not?

Problem 3: Theory-applications split in a course. (Courtesy of Stephen Boyd)
A professor teaches a course with 24 lectures, labeled $i = 1, \ldots, 24$. The course involves some interesting theoretical topics, and many practical applications of the theory. The professor must decide how to split each lecture between theory and applications. Let T_i and A_i denote the fraction of the ith lecture devoted to theory and applications, for $i = 1, \ldots, 24$. (We have $T_i \geq 0$, $A_i \geq 0$, and $T_i + A_i = 1$.)
A certain amount of theory has to be covered before the applications can be taught. We model this in a crude way as

$$A_1 + \cdots + A_i \leq \phi(T_1 + \cdots + T_i), \quad i = 1, \ldots, 24, \tag{3}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a given nondecreasing function. We interpret $\phi(u)$ as the cumulative amount of applications that can be covered, when the cumulative amount of theory covered is u. We will use the simple form $\phi(u) = a \max\{0, u - b\}$ with $a, b > 0$, which means that no applications can be covered until b lectures of the theory is covered; after that, each lecture of theory covered opens the possibility of covering a lectures on applications.
The theory-applications split affects the emotional state of students differently. We let s_i denote the emotional state of a student after lecture i, with $s_i = 0$ meaning neutral, $s_i > 0$ meaning happy, and $s_i < 0$ meaning unhappy. Careful studies have shown that s_i evolves via a linear recursion (dynamics)

$$s_i = (1 - \theta)s_{i-1} + \theta(\alpha T_i + \beta A_i), \quad i = 1, \ldots, 24,$$

with $s_0 = 0$. Here α and β are parameters (naturally interpreted as how much the student likes or dislikes theory and applications, respectively), and $\theta \in [0, 1]$ gives the emotional volatility of the student (i.e., how quickly he or she reacts to the content of recent lectures).

The student's *cumulative emotional state* (CES) is by definition $s_1 + \cdots + s_{24}$. This is a measure of his/her overall happiness throughout the semester.
Now consider a specific instance of the problem, with course material parameters $a = 2$, $b = 3$, and three groups of students, with emotional dynamics parameters given as follows:

	Group 1	Group 2	Group 3
θ	0.05	0.1	0.3
α	-0.1	0.8	-0.3
β	1.4	-0.3	0.7

Your job is to plan (four different) theory-applications splits that respectively maximize the CES of the first group, the CES of the second group, the CES of the third group, and, finally, the minimum of the cumulative emotional states of all three groups. (Hint: you would need to appropriately reformulate constraint (3) to end up with a convex optimization problem.) Report the numerical values of the CES for each group, for each of the four theory-applications splits (i.e., fill out the following table):

	Group 1	Group 2	Group 3
Plan 1			
Plan 2			
Plan 3			
Plan 4			

For each of the four plans, plot T_i as well as the emotional state s_i for all three groups, versus i. (So you should have four figures with four curves on each.) These plots show you how the emotional states of the students change as the amount of theory varies.

Problem 4: Consider the function

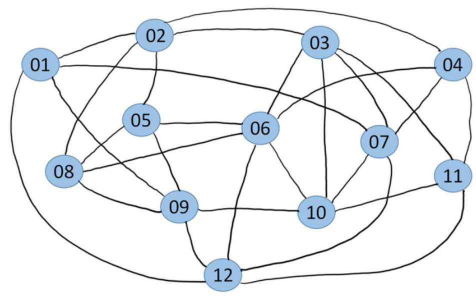
$$f(x) = x_1^2 - x_1x_2 + 2ax_1x_3 + \frac{1}{5}x_1x_4 + x_2^2 + 2bx_2x_3 + 2cx_2x_4 + x_3^2 - \frac{3}{5}x_3x_4 + x_4^2,$$

where $a, b, c \in \mathbb{R}$ are unknown parameters and $x = (x_1, x_2, x_3, x_4)^T$. Assuming f has the point $(1, 2, 2, 3)^T$ in its 10-sublevel set, use CVX to tell us the largest value of c (to three significant digits) for which f can be

(a) convex,

(b) quasiconvex. (Hint: To examine quasiconvexity of this function for parameter values for which it is not convex, consider $f(0)$, $f(\hat{x})$, and $f(-\hat{x})$ for some appropriate $\hat{x} \in \mathbb{R}^4$.)

Problem 5: Let's finish where we started. In the very first lecture, we tried to avoid "overlapping finals" by finding the largest stable set (aka independent set) in this graph:



We found one of size 5, but at the time we couldn't show that 5 was the best possible.

(a) We have seen in lecture that the optimal value of the following LP gives an upper bound on the stability number of a graph $G(V, E)$ with vertex set V of size n and edge set E:

$$\begin{aligned} \max \sum_{i=1}^{n} x_i & \\ \text{s.t. } x_i + x_j \leq 1, & \text{ if } (i,j) \in E \\ x_i \in [0,1], & \ i = 1, \ldots, n. \end{aligned} \tag{4}$$

Without solving this LP, tell us why the optimal value will not be good enough to show that there are no stable sets of size larger than 5 in the graph above.

(b) Show that our graph has no stable set of size larger than 5 by solving an SDP.

(c) Write down a new LP that still produces a valid upper bound on the stable set number of a given graph and succeeds in showing that for our particular graph there are no stable sets of size larger than 5. (Hint: Try to add additional constraints to (4) by considering triples of vertices.)

You can copy the adjacency matrix of our graph from here:

$$A_G = \begin{bmatrix} 0&1&0&0&0&0&1&0&1&0&0&1 \\ 1&0&1&1&1&0&0&1&0&0&0&0 \\ 0&1&0&0&0&1&1&0&0&1&1&0 \\ 0&1&0&0&0&1&1&0&0&0&1&0 \\ 0&1&0&0&0&1&0&1&1&0&0&0 \\ 0&0&1&1&1&0&0&1&0&1&0&1 \\ 1&0&1&1&0&0&0&0&0&1&0&1 \\ 0&1&0&0&1&1&0&0&1&0&0&0 \\ 1&0&0&0&1&0&0&1&0&1&0&1 \\ 0&0&1&0&0&1&1&0&1&0&1&0 \\ 0&0&1&1&0&0&0&0&0&1&0&1 \\ 1&0&0&0&0&1&1&0&1&0&1&0 \end{bmatrix}.$$