

Instructor: A.A. Ahmadi

TAs: B. El Khadir, G. Hall, J. Ye, J. Zhang, K. Wang, Z. Li

Due at 1:30pm in class, on September 29th, 2016

For all problems that use MATLAB, please include your code.

**Objectives of the homework:** The goal of this homework is to give you some practice with modeling and basic use of MATLAB, as well as refresh your memory on some concepts from multivariate calculus and linear algebra that are going to come up in the course.

**Problem 1: A new challenger**

Over the last few weeks, Samsung has gained much negative publicity when many of its Note 7 phones were reported to have exploded while charging. This prompted a worldwide recall for all its Galaxy Note smartphones. Not to be outdone, Apple announced the iPhone 7, which among other things, removed the headphone jack, also granting itself much negative publicity.

You are the director of marketing for the Widget Company, a phone manufacturer who has been in Apple's and Samsung's shadow for a long time. It is your chance now to take advantage of all this bad publicity. You are given a budget of 1 million dollars and would like to advertise your YoloSwag smartphones as widely as possible. The following are some strategies you can consider.

- Advertise on the big screen at Times Square in New York. If your video advertisement lasts  $x$  seconds, then the cost to you will be  $3x^2$  thousand dollars. However, you will get  $2x$  thousand customers to buy your phones.
- Hold up to a maximum of three news conferences. Each time you can attract 15 thousand customers and it will cost you 200 thousand dollars.
- Provide discounts for your YoloSwag smartphones. Of course, it would incur some loss, which should be compensated from your budget. If your loss is  $w$  thousand dollars, then you get  $0.05w$  thousand customers.
- Rename the smartphone. This can be done for free and will net 2 thousand customers.

- You can assume that different strategies will attract a completely separate group of customers.

Using only linear and/or quadratic functions, write down an optimization problem that finds a strategy attracting the maximum number of costumers. Clearly specify your decision variables, constraints, and objective function. (You don't need to solve the problem).

## Problem 2: Gradients and Hessians

1. Let  $f(x_1, x_2) = x_1^4 - x_1^3x_2 - x_1^2x_2^2 - x_1x_2^3 + x_2^4$ .

(a) Compute the gradient  $\nabla f$  and the Hessian  $\nabla^2 f$  of  $f$  by hand. Is the Hessian positive semidefinite, positive definite, negative definite, negative semidefinite, or indefinite at the following points:  $(1, 0)$  and  $(1, -1)$ ? Justify your answer without using MATLAB.

(b) Plot the function in MATLAB using the `ezsurf` function. Recall that to plot  $f(x, y) = x^2 - y^2$ , the MATLAB code can be the following:

```
1 syms x y; %declare variables
2 ezsurf(x^2-y^2);
```

(c) Define level sets. Plot the level sets of the function  $f$  above. Add its gradient vectors to your graph. What can you say about the orientation of the gradient vectors with respect to the level sets? We recommend you use the following `meshgrid` in MATLAB for plotting:

```
1 x = -5:0.2:5;
2 y = -5:0.2:5;
3 [x1, x2] = meshgrid(x, y);
```

(The `meshgrid` fixes a grid on which the function gets evaluated.) Additionally, you may want to use the functions `gradient`, `contour`, `eval` and `quiver` in MATLAB. (You can type in `doc contour` or `help contour`, for example, to see how these functions work.) The MATLAB code `hold on` enables you to plot multiple figures on one graph.

2. Let  $f(x_1, x_2, x_3, x_4) = x_1^4x_2 - \frac{x_3}{(1+x_2)^2} + 100x_1e^{x_3} + x_4^3$ .

(a) Compute the gradient of  $f$  and the Hessian of  $f$  using MATLAB.

(b) Is the Hessian positive semidefinite, positive definite, negative definite, negative semidefinite, or indefinite at the following points:  $(1, 1, -5, 0)$ ,  $(1, 1, -5, 2)$  and  $(1, 1, 1, 2)$ ? Justify numerically by using the `eig` function in MATLAB that returns the eigenvalues of a matrix.

### Problem 3: Taylor expansions

1. Consider the function  $f(x) = xe^x$ .

(a) Compute the derivatives of  $f$  up to the third order and write the 0<sup>th</sup>, 1<sup>st</sup>, 2<sup>nd</sup> and 3<sup>rd</sup> order Taylor approximation of  $f$  around  $x = 0$ .

(b) Plot  $f$  as well as its Taylor approximations computed above on the same graph but with different colors. You can use `plot`, `ezplot`, or `fplot` to do this.

(c) Plot  $\frac{|f-f_3|}{|x|^3}$ , where  $f_3$  denotes the 3<sup>rd</sup> Taylor approximation of  $f$  around  $x = 0$ . Does this agree with what Taylor's theorem tells you? (See pages 74-75 of the CZ book or Lecture 2 in the notes.) Explain.

2. Let  $f(x_1, x_2) = (x_1 + x_2)^2 - (x_1 - x_2)^4$ . Derive the 1<sup>st</sup> and 2<sup>nd</sup> order Taylor approximations of  $f$  around point  $(1, 2)$ . On the square  $\{(x_1, x_2) : 0 \leq x_1 \leq 2, 1 \leq x_2 \leq 3\}$ , plot  $f$  and the hyperplane corresponding to its first order Taylor approximation on to the same graph; then plot  $f$  and the quadratic function corresponding to the second order Taylor approximation on to the same graph.

### Problem 4: Unit spheres of different norms

The unit sphere of a norm  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is defined as its 1-level set, i.e.,  $\{x \in \mathbb{R}^n \mid f(x) = 1\}$ . Draw the unit sphere of the 1-norm, the 2-norm, and the  $\infty$ -norm on the plane (i.e., when  $n = 2$ ) by hand. In other words, if  $x = (x_1, x_2)$ , you have to plot the following sets:

$$\begin{aligned} S_1 &= \{(x_1, x_2) \in \mathbb{R}^2 \mid \|x\|_1 = 1\} \\ S_2 &= \{(x_1, x_2) \in \mathbb{R}^2 \mid \|x\|_2 = 1\} \\ S_\infty &= \{(x_1, x_2) \in \mathbb{R}^2 \mid \|x\|_\infty = 1\}. \end{aligned}$$

Show your work.

### Problem 5: Perfect numbers, roses, and practice with for loops

A positive integer is said to be *perfect* if it equals the sum of its divisors, excluding itself. For example, the number 6 is perfect because these divisors for 6 are 1, 2, 3 and we have  $1 + 2 + 3 = 6$ . The number 10 is not perfect since  $1 + 2 + 5 = 8 \neq 10$ .

1. You have been thinking about buying red roses for the person you love. You want the number of roses that you give to be perfect. This would convey the romantic message

that (s)he is perfect in your eyes. Suppose you are willing to buy up to 1000 roses (yes, love can make you do strange things), write a MATLAB code that tells you all your options. (To get credit, your code should produce the result. You may want to use the MATLAB function `mod`.)

2. In many cultures around the world, it is customary to give flowers in odd numbers. In Russia, e.g., an even number of flowers is usually only given at funerals. In the Persian culture, one gives an odd number of flowers to let the receiver know that there will be a future time with another odd number of flowers to “even up” the flowers he/she just received. Suppose you want to give your loved one a number of red roses which is both odd and perfect. What would be the smallest such number? (It is OK to give the best lower bound achieved after 10 minutes on your machine.)