ORF 363/COS 323	Problem set 7	Fall 2016, Princeton University
Instructor: A.A. Ahmadi		
TAs: B. El Khadir, G. Hall, Z. Li, K. Wang, J. Ye, J. Zhang		
Due at 1:30pm in class, on December 13, 2016		

Problem 1: To sleep or not to sleep? A student is taking a 48-hour take-home exam and would like to understand how his/her grade will be affected by the number of hours he/she devotes to sleep. Based on past surveys, the instructor has provided the class with a set of data points (t_i, y_i), where y_i is the average grade (out of 100) of previous students who slept t_i hours during the course of the exam. These data points, for $i = 1, \ldots, 7$, are as follows:

$$\{(0, 67), (8, 88), (14, 90), (24, 85), (32, 68), (40, 57), (48, 0)\}.$$

(a) Statisticians have shown that the relationship between y and t is well described by a quadratic function. Your job is to fit a quadratic polynomial

$$p(t) = c_2 t^2 + c_1 t + c_0$$

to the data given above that solves the following optimization problem:

$$\underset{c_0, c_1, c_2}{\text{minimize}} \quad \sum_{i=1}^{7} (p(t_i) - y_i)^2. \tag{1}$$

This is a least squares problem. It can be written as the minimization of a quadratic function

$$\frac{1}{2} c^T Q c - b^T c + r,$$

where Q is a 3×3 matrix, $b \in \mathbb{R}^3$, $r \in \mathbb{R}$, and $c = (c_0, c_1, c_2)^T$. Solve problem (1) by applying the conjugate gradient algorithm. Start the algorithm at the point $(0, 0, 0)^T$ and stop the iterations when the 2-norm of the gradient of the objective function becomes smaller than 10^{-5}. Report the optimal value and optimal solution of (1) and plot the optimal quadratic polynomial along with the data points in the range $t \in [0, 48]$. How many iterations did your algorithm run for? What is the condition number of Q?

(b) Unless there is some bribery going on, a student who sleeps 48 hours during the exam deserves nothing but a zero. There is no guarantee that the least squares fit will respect this feature even though our last data point is $(48, 0)$. Use CVX to resolve problem (1) subject to the following constraint:

$$p(48) = 0. \tag{2}$$

Report the new optimal value, and plot the new optimal quadratic polynomial along with the data points in the range $t \in [0, 48]$. Based on the curve you obtain, what is optimal number of hours for the student to sleep (to one significant digit)?

Problem 2: Radiation treatment planning (from [1])

In radiation treatment, radiation is delivered to a patient, with the goal of killing or damaging the cells in a tumor, while carrying out minimal damage to other tissue. The radiation is delivered in beams, each of which has a known pattern; the level of each beam can be adjusted. (In most cases multiple beams are delivered at the same time, in one 'shot', with the treatment organized as a sequence of 'shots'.) We let b_j denote the level of beam j, for $j = 1, \ldots, n$. These must satisfy $0 \leq b_j \leq B^{\max}$, where $B^{\max}$ is the maximum possible beam level. The exposure area is divided into m voxels, labeled $i = 1, \ldots, m$. The dose d_i delivered to voxel i is linear in the beam levels, i.e., $d_i = \sum_{j=1}^{n} A_{ij} b_j$. Here $A \in \mathbb{R}_+^{m \times n}$ is a (known) matrix that characterizes the beam patterns. We now describe a simple radiation treatment planning problem.

A (known) subset of the voxels, $\mathcal{T} \subset \{1, \ldots, m\}$, corresponds to the tumor or target region. We require that a minimum radiation dose D^{target} be administered to each tumor voxel, i.e., $d_i \geq D^{\text{target}}$ for $i \in \mathcal{T}$. For all other voxels, we would like to have $d_i \leq D^{\text{other}}$, where D^{other} is a desired maximum dose for non-target voxels. This is generally not feasible, so instead we settle for minimizing the penalty

$$E = \sum_{i \notin \mathcal{T}} (d_i - D^{\text{other}})_+,$$

where $(\cdot)_+$ denotes the nonnegative part of its argument (i.e., $(z)_+ = \max\{0, z\}$). We can interpret E as the total nontarget excess dose.

1. Show that the treatment planning problem is a linear program. The optimization variable is $b \in \mathbb{R}^n$; the problem data are $B^{\max}$, A, $\mathcal{T}$, D^{target}, and D^{other}.

2. Solve the problem instance with data generated by the file `treatment_planning_data.m`. Here we have split the matrix A into `Atarget`, which contains the rows corresponding to the target voxels, and `Aother`, which contains the rows corresponding to other voxels. Plot the dose histogram for the target voxels, and also for the other voxels. (You can use the MATLAB function `hist` to plot histograms.) Make a brief comment on what you see. *Remark:* The beam pattern matrix in this problem instance is randomly generated, but similar results would be obtained with realistic data.

Problem 3 (from [2]):
Find necessary and sufficient conditions on $a, b \in \mathbb{R}$ under which the linear program

$$\begin{aligned} \underset{x,y}{\text{maximize}} \quad & x + y \\ \text{subject to} \quad & ax + by \leq 1 \\ & x, y \geq 0 \end{aligned}$$

(a) is infeasible,

(b) is unbounded,

(c) has a unique optimal solution.

Problem 4: Detecting uniqueness of solutions to an LP
Consider the following LP:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0, \end{aligned}$$

where A is $m \times n$.

(a) Suppose the feasible set is nonempty and bounded. Describe an algorithm for deciding whether the optimal solution to the LP is unique. (Hint: you are allowed to solve multiple LPs.)

(b) Give an example of A, b, c, with $c \neq 0$, such that the optimal solution to the resulting LP is not unique.

Problem 5: Existence of rational solutions to an LP
Consider a linear program in standard form:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0, \end{aligned}$$

where A is an $m \times n$ matrix. Suppose for simplicity that the feasible set is nonempty and bounded. Based on topics covered in this class, show that if the entries of A, b, c are rational,[1] then there exists at least one optimal solution x^* whose entries are all rational.

[1] Recall that a real number is rational if it is the ratio of two integers (e.g., $\frac{7}{9}$).

References

[1] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge university press, 2009. Additional Exercises. Courtesy of Stephen Boyd.

[2] S. Dasgupta, C.H. Papadimitriou, and U. Vazirani, *Algorithms*. McGraw-Hill, Inc., 2006