ORF 363/COS 323	Problem set 1	Fall 2017, Princeton University
Instructor: A.A. Ahmadi		
TAs: B. El Khadir, C. Dibek, G. Hall, J. Zhang, J. Ye, S. Uysal		
Due at 1:30pm in class, on September 28th, 2017		

For all problems that use MATLAB, please include your code.

Problem 1: Optimize for happiness!

We all face constrained optimization every day even if you don't actively realize it. You by now should have realized that you have not been endowed by infinite time, money, and energy, and that this affects your decision making. Let's get a sense of how. Suppose you have 24 hours in a day, with 50 dollars and 10 units of energy (which we will call Red Bull Equivalents or RBEs). We want to maximize our happiness (in utils) over the following options:

- Do your homework. This will take exactly 2 hours at the cost of 2 RBEs, and grant 2 utils.
- Go to your part-time job. You can work for any amount of time up to 8 hours, using (proportional to) 1 RBE per hour and gaining (proportional to) 10 dollars per hour.
- Everyone has to eat exactly twice. The options are either the eating club, which takes half an hour, granting 2 utils and 3 RBEs; or to eat out for an hour at a cost of 10 dollars, granting 4 utils and 1 RBE.
- Go to New York. This will take 6 hours, 60 dollars, and 6 RBEs, and grant 8 utils.
- Sleep. Sleep as much as you want, get 1 RBE per hour of sleep.
- Party it up. For 2 hours, 20 dollars, and 2 RBEs, get 3 utils.
- Drink some Red Bull. 1 RBE (big surprise) for 2 dollars. But don't drink more than 2 in one day.

Write down the optimization problem you are facing using only linear and/or quadratic equality and inequality constraints and a linear or quadratic objective function. Clearly specify your decision variables, constraints, and objective function. (You don't need to solve the problem).

Problem 2: Gradients and Hessians

1. Let $f(x_1, x_2) = x_1^4 + x_2^4 - x_1 x_2^3 - x_1^2 x_2^2 + x_1^3 + x_2^3$.

   (a) Compute the gradient ∇f and the Hessian $\nabla^2 f$ of f by hand. Is the Hessian positive semidefinite, positive definite, negative definite, negative semidefinite, or indefinite at the following points: $(1, 0)^T$ and $(1, 1)^T$? Justify your answer without using MATLAB.

   (b) Plot the function in MATLAB using the `ezsurf` function. Recall that to plot $f(x, y) = x^2 - y^2$, the MATLAB code can be the following:

```
syms x y; %declare variables
ezsurf(x^2-y^2);
```

   (c) Define level sets. Plot the level sets of the function f above. Add its gradient vectors to your graph. What can you say about the orientation of the gradient vectors with respect to the level sets? We recommend you use the following meshgrid in MATLAB for plotting:

```
x =-5:0.2:5;
y = -5:0.2:5;
[x1,x2]=meshgrid(x,y);
```

   (The meshgrid fixes a grid on which the function gets evaluated.) Additionally, you may want to use the functions `gradient`, `contour`, `eval` and `quiver` in MATLAB. See the documentation for examples on how to use these functions by typing in `doc contour` or `help contour`. The MATLAB code `hold on` enables you to plot multiple figures one one graph.

2. Let $f(x_1, x_2, x_3, x_4) = x_1^4 \cdot x_2 - x_3 \cdot \frac{1}{(1+x_2)^2} + 100 \cdot x_1 \cdot e^{x_3} + x_4^3$.

   (a) Compute the gradient of f and the Hessian of f using MATLAB.

   (b) Is the Hessian positive semidefinite, positive definite, negative definite, negative semidefinite, or indefinite at the following points: $(1, 1, -5, 0)^T$, $(1, 1, -5, 2)^T$ and $(1, 1, 1, 2)^T$? Justify numerically by using the `eig` function in Matlab that returns the eigenvalues of a matrix.

Problem 3: Taylor expansions

1. Consider the function $f(x) = xcos(x)$.
(a) Compute the derivatives of f up to the third order and write the 0^{th}, 1^{st}, 2^{nd} and 3^{rd} Taylor approximations of f around $x = 0$.
(b) Plot f as well as its Taylor approximations computed above on the same graph but with different colors. You can use `plot`, `ezplot`, or `fplot` to do this.
(c) Plot $\frac{|f-f_3|}{|x|^3}$, where f_3 denotes the 3^{rd} Taylor approximation of f around $x = 0$. Does this agree with what Taylor's theorem tells you? (See pages 74-75 of the CZ book or Lecture 2 in the notes.) Explain.

2. Let $f(x_1, x_2) = (x_1 + x_2)^2 - (x_1 - x_2)^5$. Derive the 1^{st} order and 2^{nd} order Taylor approximations of f around point $(2, 1)$ by hand. Then use MATLAB to plot f and the hyperplane corresponding to its first order Taylor approximation on the same graph; then plot f and the quadratic function corresponding to the second order Taylor approximation on to the same graph.

Problem 4: Unit spheres of different norms

The unit sphere of a norm $f : \mathbb{R}^n \to \mathbb{R}$ is defined as its 1-level set, i.e., $\{x \in \mathbb{R}^n \mid f(x) = 1\}$. Draw the unit sphere of the 1-norm, the 2-norm, and the ∞-norm on the plane (i.e., when $n = 2$) by hand. In other words, if $x = (x_1, x_2)$, you have to plot the following sets:

$$S_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid ||x||_1 = 1\}$$
$$S_2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid ||x||_2 = 1\}$$
$$S_\infty = \{(x_1, x_2) \in \mathbb{R}^2 \mid ||x||_\infty = 1\}.$$

Show your work.

Problem 5: Perfect numbers, roses, and practice with for loops

A positive integer is said to be *perfect* if it equals the sum of its divisors, excluding itself. For example, the number 6 is perfect because these divisors for 6 are $1, 2, 3$ and we have $1 + 2 + 3 = 6$. The number 10 is not perfect since $1 + 2 + 5 = 8 \neq 10$.

1. You have been thinking about buying red roses for the person you love. You want the number of roses that you give to be perfect. This would convey the romantic message

that (s)he is perfect in your eyes. Suppose you are willing to buy up to 1000 roses (yes, love can make you do strange things), write a MATLAB code that tells you all your options. (To get credit, your code should produce the result. You may want to use the MATLAB function `mod`.)

2. In many cultures around the world, it is customary to give flowers in odd numbers. In Russia, e.g., an even number of flowers is usually only given at funerals. In the Persian culture, one gives an odd number of flowers to let the receiver know that there will be a future time with another odd number of flowers to "even up" the flowers he/she just received. Suppose you want to give your loved one a number of red roses which is both odd and perfect. What would be the smallest such number? (It is OK to give the best lower bound achieved after 10 minutes on your machine.)