ORF 363/COS 323	PROBLEM SET 3	Fall 2017, Princeton University
Instructor: A.A. Ahmadi		
TAs: B. El Khadir, C. Dibek, G. Hall, J. Zhang, J. Ye, S. Uysal		
Due at 1:30pm in class, on October 19th, 2017		

For all problems that use MATLAB, please include your code.

Problem 1: Find the range of values of the parameter α for which the following function is concave:

$$f(x_1, x_2, x_3) = 2x_1x_3 + 4x_2x_3 - x_1^2 - 2x_2^2 - 3x_3^2 - 2\alpha x_1x_2.$$

Hint: If you need to compute a 3×3 determinant, you can either do it by hand or use the MATLAB function `det`.

Problem 2: Minimizers of convex problems.
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $\Omega \subseteq \mathbb{R}^n$ be a convex set. Show that the set of minimizers of f over Ω is convex (i.e., a convex set).

Problem 3: Consider a homogeneous polynomial $p(x) = p(x_1, \dots, x_n)$ of degree $d \geq 2$. Show that if p is convex, then it is nonnegative; i.e., $p(x) \geq 0$, for all $x \in \mathbb{R}^n$. (Note: A polynomial is homogeneous of degree d if all of its monomials have degree exactly d. The degree of a monomial $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ is equal to $\alpha_1 + \cdots + \alpha_n$.)

Problem 4: Regression with different penalties using convex optimization
We are given 7 data points (x_i, y_i) in $\mathbb{R}^2$: $\{(0,0), (1,3), (2,7), (3,-1), (4,0), (5,5), (6,10)\}$. We would like to fit a cubic polynomial

$$p(x) = c_3x^3 + c_2x^2 + c_1x + c_0$$

to our data to minimize the so-called L^2-error :

$$\underset{c_0,\dots,c_3}{\text{minimize}} \quad \sum_{i=1}^{7} (p(x_i) - y_i)^2, \tag{1}$$

1. (a) Confirm that problem (1) is a least squares problem by writing it as the problem of minimizing $||Ac - b||^2$ over $c = (c_0, \ldots, c_3)^T$, for some appropriate matrix A and vector b. Find the optimal solution and the optimal value using the closed-from solution we derived in class for the least squares problem.

   (b) Solve problem (1) in CVX and compare the answer with what you got in part (a).

2. Now consider two other types of error:

   - the L^1-error:

   $$\underset{c_0,\ldots,c_3}{\text{minimize}} \quad \sum_{i=1}^{7} |p(x_i) - y_i|, \tag{2}$$

   - and the L^∞-error :

   $$\underset{c_0,\ldots,c_3}{\text{minimize}} \quad \max_{i=1,\ldots,7} |p(x_i) - y_i|. \tag{3}$$

   Although these problems no longer admit a closed-form solution, we can just as easily solve them using convex optimization.

   (a) Show that problems (2) and (3) are convex optimization problems. Solve them using CVX and report their optimal values and optimal solutions.

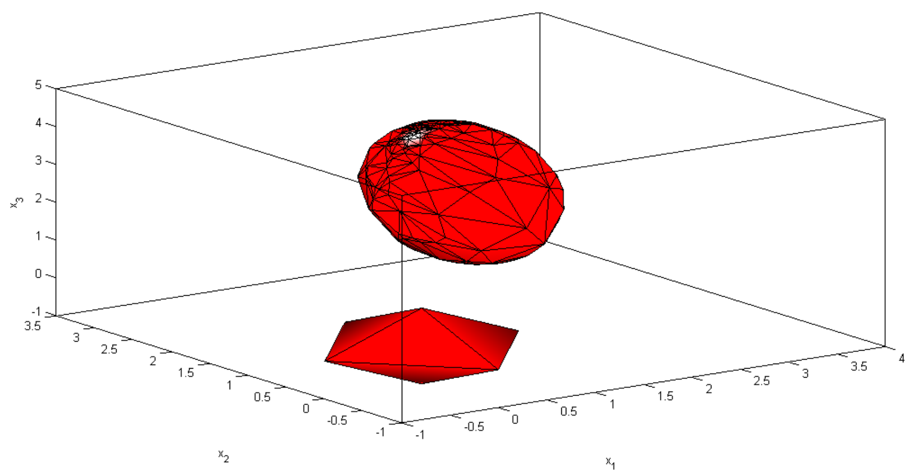
   (b) Generate a single plot with the seven data points, as well as the three cubic polynomials that are solutions to problems (1), (2), and (3). Let the x axis run from -1 to 7 and label your plots clearly. You may want to use the `hold on` command to display multiple plots on the same figure.

Problem 5: The distance between two convex sets

The distance between two sets $S_1, S_2 \subset \mathbb{R}^n$ is the closest distance between any two points one taken from each set. Solve a convex optimization problem using CVX that finds the distance between the unit ball of the L^1 norm in $\mathbb{R}^3$ (i.e., the set $S_1 = \{x \in \mathbb{R}^3 | \; ||x||_1 \leq 1\}$), and the ellipsoid $S_2 = \{x \in \mathbb{R}^3 | \; (x - x_c)^T P (x - x_c) \leq 1\}$, where $x_c = (2, 2, 2)^T$ and

$$P = \begin{pmatrix} 1 & -0.6 & 0.2 \\ -0.6 & 2.6 & 0.6 \\ 0.2 & 0.6 & 0.4 \end{pmatrix}.$$

Report this distance as well as the coordinates of two points that achieve it.


5
4
3
2
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-1
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x_1