ORF 363/COS 323	PROBLEM SET 1	Fall 2023, Princeton University
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Due on Thuesday, September 19, 2023, at 1:30pm EST, on Gradescope		

For all problems that involve coding, please include your code. Our instructions are given with MATLAB in mind (recommended language), but you are free to use a different language.

Problem1: Optimizing Cub Club

The coaches of the Princeton University varsity tennis teams run a tennis program for top juniors in the area who aspire to play college tennis (true story). The program is called Cub Club and currently has 12 sessions per month. The number of players in each session is capped at 24. This is because Princeton has only six varsity courts (either indoors or outdoors) and the coaches run drills that require a maximum of four players per court.

Assume that there are currently 40 players qualified to participate in Cub Club, each labeled with one of three skill levels L_1, L_2, L_3 by the coaches. At the beginning of each month, each player sends an email to the coaches specifying a subset of the 12 sessions that they request to participate in for that month.

1. Formulate an optimization problem for the coaches that would determine an optimal assignment of players to sessions in such a way that the total number of unmet requests[1] is minimized and such that each session has at least 5 players of each skill level. Clearly specify your decision variables, objective function, and constraints. Write your constraints in terms of affine/quadratic equations or inequalities only.

2. Formulate another optimization problem with affine/quadratic constraints and objective function whose optimal value would allow you to determine whether the optimal assignment of players to sessions in the previous part is unique. (Hint: consider introducing a copy of each decision variable.)

[1] This means the total number of sessions across players that were requested but cannot be accommodated.

Problem 2: Perfect numbers, roses, and practice with for loops

A positive integer is said to be *perfect* if it equals the sum of its divisors, excluding itself. For example, the number 6 is perfect because these divisors for 6 are $1, 2, 3$ and we have $1 + 2 + 3 = 6$. The number 10 is not perfect since $1 + 2 + 5 = 8 \neq 10$.

1. You have been thinking about buying red roses for the person you love. You want the number of roses that you give to be perfect. This would convey the romantic message that (s)he is perfect in your eyes. Suppose you are willing to buy up to 1000 roses (yes, love can make you do strange things). Write a code that tells you all your options. (To get credit, your code should produce the result. You may want to use the MATLAB function `mod`.)

2. In many cultures around the world, it is customary to give flowers in odd numbers. In Russia, e.g., an even number of flowers is usually only given at funerals. In the Persian culture, one gives an odd number of flowers to let the receiver know that there will be a future time with another odd number of flowers to "even up" the flowers he/she just received. Suppose you want to give your loved one a number of red roses which is both odd and perfect. What would be the smallest such number? (It is OK to give the best lower bound achieved after 10 minutes on your machine.)

Problem 3: Taylor expansions

1. Consider the function $f(x) = xe^x$.

   (a) Compute the derivatives of f up to the third order and write the 0^{th}, 1^{st}, 2^{nd} and 3^{rd} Taylor approximations of f around $x = 0$.

   (b) Plot f as well as its Taylor approximations computed above on the same graph but with different colors. You can use `plot`, `ezplot`, or `fplot` to do this.

   (c) Plot $\frac{|f-f_3|}{|x|^3}$, where f_3 denotes the 3^{rd} Taylor approximation of f around $x = 0$. Does this agree with what Version 1 of Taylor's theorem from Lecture 2 tells you?

2. Let $f(x_1, x_2) = (x_1 + x_2)^2 - (x_1 - x_2)^5$. Derive the 1^{st} order and 2^{nd} order Taylor approximations of f around point $(2, 1)$ by hand. Then use MATLAB to plot f and the hyperplane corresponding to its first order Taylor approximation on the same graph; then plot f and the quadratic function corresponding to the second order Taylor approximation on to the same graph.

Problem 4: Gradients and Hessians

1. Let $f(x_1, x_2) = x_1^4 + x_2^4 - x_1 x_2^3 - x_1^2 x_2^2 + x_1^3 + x_2^3$.

   (a) Compute the gradient ∇f and the Hessian $\nabla^2 f$ of f by hand. Is the Hessian positive semidefinite, positive definite, negative definite, negative semidefinite, or indefinite at the following points: $(1,0)^T$ and $(1,1)^T$? Justify your answer without using MATLAB.

   (b) Plot the function in MATLAB using the `ezsurf` function. Recall that to plot $f(x,y) = x^2 - y^2$, the MATLAB code can be the following:

```
syms x y; %declare variables
ezsurf(x^2-y^2);
```

   (c) Define level sets. Plot the level sets of the function f above. Add its gradient vectors to your graph. What can you say about the orientation of the gradient vectors with respect to the level sets? We recommend you use the following meshgrid in MATLAB for plotting:

```
 x =-5:0.2:5;
 y = -5:0.2:5;
 [x1,x2]=meshgrid(x,y);
```

   (The meshgrid fixes a grid on which the function gets evaluated.) Additionally, you may want to use the functions `gradient`, `contour`, `eval` and `quiver` in MATLAB. See the documentation for examples on how to use these functions by typing in `doc contour` or `help contour`. The MATLAB code `hold on` enables you to plot multiple figures one one graph.

2. Let $f(x_1, x_2, x_3, x_4) = x_1^4 \cdot x_2 - x_3 \cdot \frac{1}{(1+x_2)^2} + 100 \cdot x_1 \cdot e^{x_3} + x_4^3$.

   (a) Compute the gradient of f and the Hessian of f using MATLAB.

   (b) Is the Hessian positive semidefinite, positive definite, negative definite, negative semidefinite, or indefinite at the following points: $(1,1,-5,0)^T$, $(1,1,-5,2)^T$ and $(1,1,1,2)^T$? Justify numerically by using the `eig` function in Matlab that returns the eigenvalues of a matrix.