Name:________________________________

Princeton University

ORF 523
Midterm Exam # 2, Spring 2016

April 7, 2016, from 1:30 pm to 2:50 pm

Instructor: A.A. Ahmadi

AI: G. Hall

PLEASE DO NOT TURN THIS PAGE. START THE EXAM ONLY WHEN YOU ARE INSTRUCTED TO DO SO.

1. You are allowed a single sheet of A4 paper, double sided, hand-written or typed.
2. No electronic devices are allowed (e.g., cell phones, calculators, laptops, electronic cigarettes, electronic marijuana, etc.).
3. All questions should be answered in the booklet. Please write out and sign the following pledge on the booklet: "I pledge my honor that I have not violated the honor code during this examination."
4. You are allowed to cite results shown in lecture or problem sets without proof, but nothing else.

Throughout, the notation $S^{n\times n}$ denotes the set of real, symmetric $n \times n$ matrices.

Question 1 (15 pts): Given an example of an SDP with rational data whose optimal solutions are all irrational.

Question 2 (20 pts): Consider the SDP

$$\begin{aligned} &\min_{X \in S^{3\times 3}} X_{22} \\ &\text{s.t. } X_{11} = 0,\ 2X_{13} + X_{22} = 1,\ X \succeq 0. \end{aligned}$$

What is the optimal value of this SDP? What is the optimal value of its dual? Briefly comment on your answers in view of the strong duality theorem for SDP.

Question 3 (15 pts): Let K denote the cone of positive semidefinite matrices of dimension n, i.e.,

$$K := \{A \in S^{n\times n} \mid y^T A y \geq 0, \forall y \in \mathbb{R}^n\}.$$

Define a new cone

$$K^* := \{B \in S^{n\times n} \mid \text{Tr}(AB) \geq 0, \forall A \in K\}.$$

Show that $K^* = K$.

Question 4 (25 pts): Show that the Shannon capacity of the cycle of length four is equal to 2.

Question 5 (25 pts): Consider a dynamical system $x_{k+1} = Ax_k$, where $A \in \mathbb{R}^{n\times n}$. Suppose that the spectral radius of A is strictly less than 1 and consider the set

$$\mathcal{S} := \{x \in \mathbb{R}^n \mid x^T x \leq 1\}.$$

Give an SDP-based algorithm that constructs a set $\mathcal{S}'$ such that (i) $\partial\mathcal{S} \cap \partial\mathcal{S}'$ is nonempty[1], and (ii) if $x_0 \in \mathcal{S}'$, then $x_k \in \mathcal{S}$ for all k.

[1]The symbol ∂ here denotes the boundary of a set; e.g., $\partial\mathcal{S} = \{x \in \mathbb{R}^n \mid x^T x = 1\}$.