Name:________________________________

PRINCETON UNIVERSITY

ORF 523

Final Exam, Spring 2017

WEDNESDAY, MAY 17, 9AM, TO MONDAY, MAY 22, 9AM

Instructor: A.A. Ahmadi

AIs: G. Hall, C.Y. Liu

1. Please write out and sign the following pledge on top of the first page of your exam: "I pledge my honor that I have not violated the Honor Code or the rules specified by the instructor during this assignment."

2. Don't forget to write your name. Make a copy of your solutions and keep it.

3. The assignment is not to be discussed with *anyone* except possibly the professor and the TAs. You can only ask clarification questions as public questions on Piazza. No emails.

4. You are allowed to consult the lecture notes, your own notes, the recommended textbooks of the course, the problem sets and their solutions (yours and ours), the midterm exams and their solutions (yours and ours), but *nothing else*. You can only use the Internet in case you run into problems related to MATLAB, CVX or YALMIP. (There should be no need for that either hopefully.)

5. For all problems involving MATLAB, CVX, or YALMIP, show your code. The MATLAB output that you present should come from your code.

6. The assignment is to be turned in before Monday, May 22, at 9 AM in the box for ORF 523 in Sherrerd 123. Please time stamp your exam (just write the time of drop off and sign it). If you are away, you can email a single PDF file to the instructor and the AIs.

7. Good luck!

Grading

Problem 1	20 pts	
Problem 2	20 pts	
Problem 3	20 pts	
Problem 4	20 pts	
Problem 5	20 pts	
TOTAL	100	

Problem 1: Difference of sum of squares decomposition

Prove or disprove the following statement: Any (multivariate) polynomial can be written as the difference of two sum of squares polynomials.[1]

Problem 2: Shape-constrained regression

A study has provided us with m data points $(x_i, y_i)_{i=1,\ldots,m}$, where x_i is a 2×1 vector and y_i is a scalar. These data points are known to be noisy samples of a bivariate quartic polynomial, which is nonnegative (globally) and convex when restricted to the line passing through the points $(0,1)$ and $(1,1)$.

Our goal is to find a bivariate quartic polynomial p which:

(i) satisfies the nonnegativity and convexity-along-the-line requirements,

(ii) minimizes the least absolute deviations objective function $\sum_{i=1}^m |y_i - p(x_i)|$.

Write a semidefinite program that finds such a p and carefully argue why your formulation is correct.

Problem 3: Testing copositivity

For each of the following two matrices

$$A = \begin{pmatrix} 30 & 20 & 17 & 13 \\ 22 & 13 & 0 & -6 \\ 6 & -4 & 39 & 33 \\ 11 & -3 & 28 & 41 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{pmatrix},$$

prove or disprove the claim that they are copositive.

Problem 4: Containment among polytopes

A polytope (i.e., a bounded polyhedron) P in $\mathbb{R}^n$ can be represented either through a facet description (as the feasible set of finitely many affine inequalities) or through a vertex description (as the convex hull of a finite set of points in $\mathbb{R}^n$). Given two polytopes P_1 and P_2, we would like to design an algorithm that checks if $P_1 \subseteq P_2$. There are four possibilities here based on the facet/vertex description of each polytope. Out of these four, for how many can you propose an algorithm whose worst-case running time is not exponential in the dimension

[1] Recall that we say a polynomial is a *sum of squares*, if it can be written as a sum of squares of other polynomials.

n, or the number of input facets, or the number of input vertices? (Hint: do not be overly greedy, unless you are going for fame and fortune.)

Problem 5: Containment among linear/quadratic basic semialgebraic sets
Consider the following decision problem: Given an $m \times n$ matrix A, an $m \times 1$ vector b, an $n \times n$ symmetric matrix Q, an $n \times 1$ vector d, and a scalar c (all data is assumed to be rational), test whether

$$\{x \in \mathbb{R}^n |\ Ax \leq b\} \subseteq \{x \in \mathbb{R}^n |\ x^T Q x + d^T x + c \leq 1\}.$$

Show that this problem is NP-hard.