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PRINCETON UNIVERSITY

ORF 523
Final Exam, Spring 2018

THURSDAY, MAY 10, 5PM, TO THURSDAY, MAY 17, 5PM

Instructor:
A.A. Ahmadi

AI:
B. El Khadir

1. Please write out and sign the following pledge on top of the first page of your exam: "I pledge my honor that I have not violated the Honor Code or the rules specified by the instructor during this assignment."

2. Don't forget to write your name. Make a copy of your solutions and keep it.

3. The assignment is not to be discussed with *anyone* except possibly the professor and the TAs. You can only ask clarification questions as public questions on Piazza. No emails.

4. You are allowed to consult the lecture notes, your own notes, the problem sets and their solutions (yours and ours), the midterm exams and their solutions (yours and ours), but *nothing else*. You can only use the Internet in case you run into problems related to MATLAB, CVX or YALMIP. (There should be no need for that either hopefully.)

5. For all problems involving MATLAB, CVX, or YALMIP, show your code. The MATLAB output that you present should come from your code.

6. The assignment is to be turned in before Thursday, May 17, at 5 PM in the box for ORF 523 in Sherrerd 123. Please time stamp your exam (just write the time of drop off and sign it). If you are away, you can email a single PDF file to the instructor and the AI.

7. Please be rigorous, brief, and to the point in your answers. Good luck!

Grading

Problem 1	25 pts	
Problem 2	25 pts	
Problem 3	25 pts	
Problem 4	25 pts	
TOTAL	100	

Problem 1: For a set $S \subseteq \mathbb{R}^n$, define $S^\diamondsuit$ to be the set

$$S^\diamondsuit := \{y \in \mathbb{R}^n \mid x^T y \leq 1 \ \forall x \in S\}.$$

1. Find $S^\diamondsuit$ for the following sets:

   (a) $S = \{x \in \mathbb{R}^n \mid x^T P x \leq 1\}$, where P is a positive definite matrix.

   (b) $S = \{x \in \mathbb{R}^n \mid Ax \leq 1\}$, where $A \in \mathbb{R}^{m \times n}$, and 1 is the vector of all ones. (Hint: If $S^\diamondsuit$ also happens to be a polyhedron, it is OK to represent it as the convex hull of its extreme points instead of giving its defining inequalities.)

2. Prove or disprove the following statements:

   (a) $A \subseteq B \implies B^\diamondsuit \subseteq A^\diamondsuit$.

   (b) $(A \cap B)^\diamondsuit = \text{conv}(A^\diamondsuit \cup B^\diamondsuit)$, where conv denotes the convex hull operation.

   (c) $(A \cup B)^\diamondsuit = A^\diamondsuit \cap B^\diamondsuit$.

Problem 2: Monotone regression

1. In the file `regression_data.mat`, you are given 20 points (x_i, f_i) in $\mathbb{R}^2$ where $(x_i)_{i=1,\ldots,20}$ are the entries of the vector `xvec` and $(f_i)_{i=1,\ldots,20}$ are the entries of the vector `fvec`.

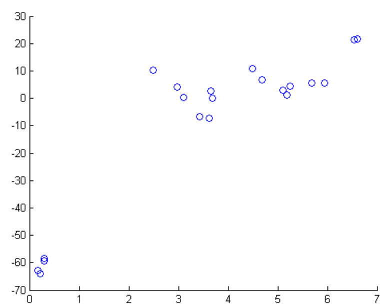

Figure 1: Figure generated by `scatter(xvec,fvec)`

The goal is to fit a polynomial of degree 7

$$p(x) = c_0 + c_1 x + \ldots c_7 x^7 \tag{1}$$

to the data to minimize the least squares error:

$$\min_{c_0,c_1,\ldots,c_7} \sum_{i=1}^{20} (p(x_i) - f_i)^2. \tag{2}$$

The data comes from noisy measurements of an unknown function that is a priori known to be nondecreasing (e.g., the number of calories you intake as a function of the number of Big Macs you eat).

(a) If the underlying function is truly monotone and the noise is not too large, one may hope that the solution to the least squares problem would automatically respect the monotonicity constraint. Solve problem (2) to see if this is the case. Plot the optimal polynomial you get and report the optimal value.

(b) Resolve (2) subject to the constraint that the polynomial (1) be nondecreasing. Plot the optimal polynomial you get and report the optimal value.

2. Monotone polynomial regression is also interesting in the multivariate setting, though no longer tractable. We say that a polynomial $p(x_1, \ldots, x_n)$ is *nondecreasing* with respect to variable[1] x_1 if

$$\forall x_1, \tilde{x}_1, x_2, \ldots, x_n \in \mathbb{R}, \quad x_1 \leq \tilde{x}_1 \implies p(x_1, x_2, \ldots, x_n) \leq p(\tilde{x}_1, x_2, \ldots, x_n).$$

Show that given a degree 5 polynomial $p(x_1, \ldots, x_n)$ with rational coefficients, the problem of testing if p is nondecreasing with respect to x_1 is NP-hard.

Problem 3: Stability of matrix products

Recall that the spectral radius $\rho(A)$ of a matrix $A \in \mathbb{R}^{n\times n}$ is the maximum of the modulus of its eigenvalues. We call a matrix "stable" if $\rho(A) < 1$. Let us call a pair of real $n \times n$ matrices $\{A_1, A_2\}$ stable if $\rho(\Sigma) < 1$, for any finite product Σ out of A_1 and A_2. (For example, Σ could be $A_2A_1, A_1A_2, A_1A_1A_2A_1$, and so on.)

Find the largest γ such that the pair

$$A_{1,\alpha} = \alpha \begin{pmatrix} -1 & -1 \\ -4 & 0 \end{pmatrix}, \qquad A_{2,\alpha} = \alpha \begin{pmatrix} 3 & 3 \\ -2 & 1 \end{pmatrix}$$

is stable for all $\alpha \in [0, \gamma)$. It is enough to find this cutoff value to three digits after the decimal point.

[1] The definition clearly applies to any other variable x_i.

Problem 4: Let

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 2 \\ -0.5 & -1 \\ 1 & -3 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, c = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, G = \frac{5}{6} \begin{pmatrix} 0.5 & 0.5 \\ -1 & 0.25 \end{pmatrix}.$$

Find the exact optimal value and optimal solution of the following linear program with an infinite number of constraints:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^T x \\ \text{s.t.} \quad & AG^k x \leq b, \ \text{for } k = 1, 2, 3, \ldots \end{aligned} \tag{3}$$

Show your reasoning carefully. (Hint: If you want to build some geometric intuition, you can use the command `plot([A*x<=b])` in YALMIP to visualize the polytope $\{x \in \mathbb{R}^n | \ Ax \leq b\}$.)