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Princeton University

ORF 523
Midterm Exam 2, Spring 2018

April 12, 2018, from 1:30 pm to 2:50 pm

Instructor:
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AI:
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PLEASE DO NOT TURN THIS PAGE. START THE EXAM ONLY WHEN YOU ARE INSTRUCTED TO DO SO.

1. You are allowed a single sheet of A4 paper, double sided, hand-written or typed.
2. No electronic devices are allowed (e.g., cell phones, calculators, laptops, etc.).
3. All questions should be answered in the booklet. Please write out and sign the following pledge on the booklet: "I pledge my honor that I have not violated the honor code during this examination."
4. You can cite results shown in lecture or on problem sets without proof.

Problem 1 (30 pts): Let G be an undirected graph with n nodes and adjacency matrix A. A *cut* in G is any labeling of the nodes by $\{0, 1\}$; the *value* of a cut is the number of edges whose endpoints are labeled differently. The following nonconvex optimization problem finds a cut of maximum value in G:

$$\begin{aligned} f^* := \max_{x \in \mathbb{R}^n} \quad & \frac{1}{4} \sum_{i,j} A_{ij}(1 - x_i x_j) \\ \text{s.t.} \quad & x_i^2 = 1, \; i = 1, \ldots, n. \end{aligned}$$

Consider the SDP

$$\begin{aligned} f^{SDP} := \min_{X \in S^{n \times n}} \quad & Tr(AX) \\ \text{s.t.} \quad & X_{ii} = 1, \; i = 1, \ldots, n, \\ & X \succeq 0. \end{aligned}$$

Show that $f^* \leq \frac{1}{4} \sum_{i,j} A_{ij} - \frac{1}{4} f^{SDP}$.

Problem 2 (40 pts): True or False? (Provide a proof or a counterexample.)

(a) If matrices $A_1, \ldots, A_m \in \mathbb{R}^{n \times n}$ are stable[1], then so is any matrix in their convex hull.

(b) If there exists a homogeneous quadratic polynomial $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(0) = 0$, $V(x) > 0, \forall x \neq 0$, and

$$V(A_i x) < V(x) \; \forall x \neq 0, \forall i \in \{1, \ldots, m\},$$

then any matrix in the convex hull of $A_1, \ldots, A_m$ is stable.

Problem 3 (30 pts): Let $c, \bar{a}_1, \ldots, \bar{a}_m \in \mathbb{R}^n$ be a set of vectors, $P_1, \ldots, P_m \in S^{n \times n}$ a set of positive definite matrices, and $b_1, \ldots, b_m \in \mathbb{R}$ a set of scalars. A *robust linear program with ellipsoidal uncertainty* is an optimization problem of the form

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^T x \\ \text{s.t.} \quad & a_i^T x \leq b_i, \forall a_i \in U_i, \forall i \in \{1, \ldots m\}, \end{aligned} \tag{1}$$

where

$$U_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \leq 1\}.$$

In other words, this is an LP where the rows of the constraint matrix are uncertain, but each known to be within an ellipsoid centered at a nominal vector. Show that problem (1) can be written as an SOCP. (Hint: you may want to use the Cauchy-Schwarz inequality somewhere in your derivation.)

[1] Recall that we say a matrix is stable if all its eigenvalues have magnitude less than one.