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PRINCETON UNIVERSITY

ORF 523

Final Exam, Spring 2019

MONDAY, MAY 6, 9AM, TO WEDNESDAY, MAY 15, 5PM

Instructor: A.A. Ahmadi

AI: C. Dibek

1. Please write out and sign the following pledge on top of the first page of your exam:
   "I pledge my honor that I have not violated the Honor Code or the rules specified by the instructor during this assignment."
2. Do not forget to write your name. Make a copy of your solutions and keep it.
3. The assignment is not to be discussed with *anyone*. You can only ask clarification questions as public (and preferably non-anonymous) questions on Piazza. No emails.
4. You are allowed to consult the lecture notes, your own notes, the problem sets and their solutions (yours and ours), the midterm exams and their solutions (yours and ours), but *nothing else*. You can only use the Internet in case you run into problems related to MATLAB or CVX. (There should be no need for that either hopefully.)
5. For all problems involving MATLAB or CVX, show your code. The MATLAB output that you present should come from your code.
6. The assignment is to be turned in before Wednesday, May 15, at 5 PM in the box for ORF 523 in Sherrerd 123. Please time stamp your exam (just write the time of drop off and sign it). If you are away, you can email a single PDF file to the instructor and the TA.
7. Some problems might be harder than others; there is no particular order. Please be rigorous, brief, and to the point in your answers. Good luck!

Grading

Problem 1	20 pts	
Problem 2	20 pts	
Problem 3	20 pts	
Problem 4	20 pts	
Problem 5	20 pts	
TOTAL	100	

Problem 1: Let $p := p(x_1, \ldots, x_n)$ be a polynomial of degree d and let $\bar{x}$ be an arbitrary point in $\mathbb{R}^n$. What is the largest value of d for which the following claim is correct?

$$\text{There is no descent direction at } \bar{x} \text{ for } p \iff \bar{x} \text{ is a local minimum for } p.$$

Justify your answer fully.

Problem 2: Consider the following decision problem:

CLOSED
Input: Vectors $a_1, \ldots, a_m, c_1, \ldots, c_r \in \mathbb{Q}^n$, scalars $b_1, \ldots, b_m, d_1, \ldots, d_r \in \mathbb{Q}$.
Question: Is the set $P = \{x \in \mathbb{R}^n \,|\, a_i^T x \leq b_i, i = 1, \ldots, m,\ c_j^T x < d_j, j = 1, \ldots, r\}$ closed?

Show that CLOSED$\in$P.

Problem 3: Consider a graph $G(V, E)$ on n vertices and let

$$\begin{aligned} \vartheta(G) \quad = \quad & \max_{X \in S^{n \times n}} && \mathrm{Tr}(JX) \\ & \text{s.t.} && X_{ij} = 0 \quad \text{if} \quad \{i, j\} \in E \\ & && \mathrm{Tr}(X) = 1, X \succeq 0, \end{aligned}$$

$$\begin{aligned} \eta_{LP}(G) \quad = \quad & \max_{x \in \mathbb{R}^n} && \sum_{i=1}^n x_i \\ & \text{s.t.} && 0 \leq x_i \leq 1 \qquad i = 1, \ldots, n \\ & && C_2, C_3, C_4, \ldots, C_{|V|}, \end{aligned}$$

where C_k contains all clique inequalities of order k, i.e. the constraints $x_{i_1} + \ldots + x_{i_k} \leq 1$ for all $\{i_1, \ldots, i_k\} \subseteq V$ defining a clique of size k. We know (from lecture and a previous problem set) that both $\vartheta(G)$ and $\eta_{LP}(G)$ provide upper bounds on the stability number of G, and that $\vartheta(G) \leq \eta_{LP}(G)$.
Show that the gap between $\vartheta(G)$ and $\eta_{LP}(G)$ can be arbitrarily large.

(Hint: First demonstrate a graph H for which $\vartheta(H) < \eta_{LP}(H)$, then see how you can construct a family of graphs using H in order to amplify the gap between these two quantities.)

Problem 4: Consider a family of decision problems indexed by a positive integer k:

RANK-k-SDP
Input: Symmetric $n \times n$ matrices $A_1, \ldots, A_m$ with entries in $\mathbb{Q}$, scalars $b_1, \ldots, b_m \in \mathbb{Q}$.
Question: Is there a real symmetric matrix X that satisfies the constraints

$$\mathrm{Tr}(A_i X) = b_i, i = 1, \ldots, m,\ X \succeq 0,\ \mathrm{rank}(X) = k?$$

Show that RANK-k-SDP is NP-hard for any integer $k \geq 1$.

(Hint: First show NP-hardness for $k = 1$, then see how you can modify your construction so that it would work for any other k.)

Problem 5: In this problem, we consider the sum of squares (SOS) relaxation for minimizing homogeneous polynomials on the sphere. First, show that the following decision problem is NP-hard: Given an n-variate degree-4 homogeneous polynomial p and an integer k, decide if the minimum of p over the unit sphere is less than k?
Consider now an instance of this problem:

$$\begin{aligned} \min_{x_1, x_2} \quad & x_1^4 - 2x_1^2x_2^2 + 4x_1x_2^3 - 7x_1^3x_2 - 3x_2^4 \\ \text{s.t.} \quad & x_1^2 + x_2^2 = 1. \end{aligned} \tag{1}$$

1. Is the objective function or the feasible set convex?
2. Denote the objective function of (1) by $p(x) := p(x_1, x_2)$. Show that any scalar γ for which
$$p(x) - \gamma(x_1^2 + x_2^2)^2$$
is SOS serves as a lower bound on the optimal value of (1). Find the largest such lower bound using CVX. (Since CVX does not allow you to impose SOS constraints, you would have to derive the underlying SDP by hand.)
3. Show that the SOS bound obtained in the previous part is tight by producing a matching upper bound (say to three digits after the decimal point). Show that if we had given you any other bivariate homogeneous polynomial of some degree $2d$ as the objective of (1), the SOS bound—obtained by finding the largest γ such that $p(x) - \gamma(x_1^2 + x_2^2)^d$ is SOS—would have still been tight.

   (Hint: You can use the following fact without proof: A univariate polynomial is nonnegative if and only if it is a sum of squares.)