ORF 523	PROBLEM SET 2	Spring 2019, Princeton University
Instructor: A.A. Ahmadi		
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Problem 1: Convex hull of a set

Show that the convex hull of a set $S \subseteq \mathbb{R}^n$ is the intersection of all convex sets that contain S.

Problem 2: True or False?

Specify whether each of the following statements is true or false and provide either a proof or a counter-example depending on your answer. Let S be a set in $\mathbb{R}^n$.

(i) If S is closed, then the convex hull of S is closed.

(ii) If S is bounded, then the convex hull of S is bounded.

(iii) If S is compact, then the convex hull of S is compact.
(You may want to use the following fact from analysis: the image of a compact set under a continuous mapping is compact.)

(iv) The sum of two quasiconvex functions is quasiconvex.

(v) A quadratic function $f(x) = x^T Qx + b^T x + c$ is convex if and only if it is quasiconvex.
(You can use the fact that f is convex if and only if $Q \succeq 0$ if you need to.)

Problem 3: Symmetries and convex optimization

(courtesy of Pablo Parrilo & Stephen Boyd)

Suppose $\mathcal{G} = \{Q_1, \ldots, Q_k\} \subseteq \mathbb{R}^{n\times n}$ is a group, i.e., closed under products and inverse. We say that the function $f : \mathbb{R}^n \to \mathbb{R}$ is *$\mathcal{G}$-invariant*, or *symmetric with respect to $\mathcal{G}$*, if $f(Q_i x) = f(x)$ holds for all x and $i = 1, \ldots, k$. We define $\bar{x} = (1/k)\sum_{i=1}^k Q_i x$, which is the average of x over its $\mathcal{G}$-orbit. We define the *fixed subspace* of $\mathcal{G}$ as

$$\mathcal{F} = \{x \mid Q_i x = x, i = 1, \ldots, k\}.$$

(a) Show that for any $x \in \mathbb{R}^n$, we have $\bar{x} \in \mathcal{F}$.

(b) Show that if $f : \mathbb{R}^n \to \mathbb{R}$ is convex and $\mathcal{G}$-invariant, then $f(\bar{x}) \le f(x)$.

(c) We say the optimization problem

$$\begin{aligned} &\text{minimize } f_0(x) \\ &\text{subject to } f_i(x) \leq 0, i = 1, \ldots, m \end{aligned}$$

is $\mathcal{G}$-invariant if the objective f_0 is $\mathcal{G}$-invariant and the feasible set is $\mathcal{G}$-invariant, which means

$$f_1(x) \leq 0, \ldots, f_m(x) \leq 0 \to f_1(Q_i) \leq 0, \ldots, f_m(Q_i x) \leq 0,$$

for $i = 1, \ldots, k$. Show that if the problem is convex and $\mathcal{G}$-invariant, and there exists an optimal point, then there exists an optimal point in $\mathcal{F}$. In other words, we can adjoin the equality constraints $x \in \mathcal{F}$ to the problem, without loss of generality.

(d) As an example, suppose that f is convex and symmetric, i.e., $f(Px) = f(x)$ for every permutation P. Show that if f has a minimizer, then it has a minimizer of the form $\alpha 1$. (This means to minimize f over $x \in \mathbb{R}^n$, we can just as well minimize $f(t1)$ over $t \in \mathbb{R}$.)

Problem 4: Containment among polytopes

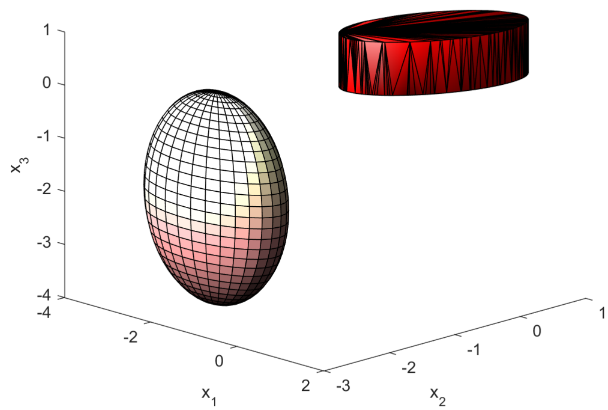
A polytope (i.e., bounded polyhedron) P in $\mathbb{R}^n$ can be represented either through a facet description (as the intersection of finitely many affine inequalities) or through a vertex description (as the convex hull of a finite set of points in $\mathbb{R}^n$). Given two polytopes P_1 and P_2, we would like to design an algorithm that checks if $P_1 \subseteq P_2$. There are four possibilities here based on the facet/vertex description of each polytope. Out of these four, for how many can you propose an algorithm whose worst-case running time is not exponential in the dimension n, or the number of input facets, or the number of input vertices? (Hint 1: you are allowed to use the fact that linear programs can be solved in polynomial time. Hint 2: don't be overly greedy, unless you are going for fame and fortune.)

Problem 5: Distance between an ellipsoid and an elliptic cylinder

The distance between two sets S_1 and S_2 is the closest distance between any two points one taken from each set.

1. Consider an elliptic cylinder S_1, given by

$$S_1 = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & 0.5 \\ x_2 & 0.5 & 1 \end{pmatrix} \succeq 0, \quad 0 \leq x_3 \leq 1 \right\},$$



and an ellipsoid S_2, given by $S_2 = \{x \in \mathbb{R}^3 \mid x^T Q x + b^T x + c \leq 1\}$, where

$$Q = \begin{pmatrix} 4/9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}, \quad b = \begin{pmatrix} 16/9 & 16 & 1 \end{pmatrix}^T, \quad c = 169/9.$$

Formulate the problem of finding the distance between S_1 and S_2 as a convex optimization problem. Use CVX to find this distance and report the value.

2. Using the commands `hold on` and `plot3`, plot on the MATLAB figure (given in `distance_computation.fig`) the two points that achieve the minimum distance computed in the previous question as well as a line segment connecting them.

3. Recall that a point x is an *extreme point* of a convex set S if it is in S and cannot be written as a convex combination of two other points in S. In other words, there does not exist $y, z \in S, y \neq x, z \neq x$, and $\lambda \in (0, 1)$ such that $\lambda y + (1 - \lambda)z = x$.

   A student who hasn't taken ORF523 claims that for any two bounded and closed convex sets S_1 and S_2 which do not intersect, there exists an extreme point in S_1 or in S_2 where the minimum distance between S_1 and S_2 is achieved. Prove the student wrong. (A correct picture is enough.)

Problem 6: Theory-applications split in a course. (Courtesy of Stephen Boyd)
A professor teaches a course with 24 lectures, labeled $i = 1, \ldots, 24$. The course involves some interesting theoretical topics, and many practical applications of the theory. The professor must decide how to split each lecture between theory and applications. Let T_i and A_i denote the fraction of the ith lecture devoted to theory and applications, for $i = 1, \ldots, 24$. (We have $T_i \geq 0$, $A_i \geq 0$, and $T_i + A_i = 1$.)
A certain amount of theory has to be covered before the applications can be taught. We model this in a crude way as

$$A_1 + \cdots + A_i \leq \phi(T_1 + \cdots + T_i), \quad i = 1, \ldots, 24, \tag{1}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a given nondecreasing function. We interpret $\phi(u)$ as the cumulative amount of applications that can be covered, when the cumulative amount of theory covered is u. We will use the simple form $\phi(u) = a \max\{0, u - b\}$ with $a, b > 0$, which means that no applications can be covered until b lectures of the theory is covered; after that, each lecture of theory covered opens the possibility of covering a lectures on applications.
The theory-applications split affects the emotional state of students differently. We let s_i denote the emotional state of a student after lecture i, with $s_i = 0$ meaning neutral, $s_i > 0$ meaning happy, and $s_i < 0$ meaning unhappy. Careful studies have shown that s_i evolves via a linear recursion (dynamics)

$$s_i = (1 - \theta)s_{i-1} + \theta(\alpha T_i + \beta A_i), \quad i = 1, \ldots, 24,$$

with $s_0 = 0$. Here α and β are parameters (naturally interpreted as how much the student likes or dislikes theory and applications, respectively), and $\theta \in [0, 1]$ gives the emotional volatility of the student (i.e., how quickly he or she reacts to the content of recent lectures). The student's *cumulative emotional state* (CES) is by definition $s_1 + \cdots + s_{24}$. This is a measure of his/her overall happiness throughout the semester.
Now consider a specific instance of the problem, with course material parameters $a = 2$, $b = 3$, and three groups of students, with emotional dynamics parameters given as follows:

	Group 1	Group 2	Group 3
θ	0.05	0.1	0.3
α	-0.1	0.8	-0.3
β	1.4	-0.3	0.7

Your job is to plan (four different) theory-applications splits that respectively maximize the CES of the first group, the CES of the second group, the CES of the third group, and, finally, the minimum of the cumulative emotional states of all three groups. Report the numerical values of the CES for each group, for each of the four theory-applications splits (i.e., fill out the following table):

	Group 1	Group 2	Group 3
Plan 1			
Plan 2			
Plan 3			
Plan 4			

For each of the four plans, plot T_i as well as the emotional state s_i for all three groups, versus i. (So you should have four figures with four curves on each.) These plots show you how the emotional states of the students change as the amount of theory varies.