

**Problem 1: Hillary or Bernie?**

You would like to use the knowledge you've acquired in optimization over the past few weeks to see if you could have predicted the outcome of each Hillary-Bernie race in the Democratic primaries. To make things easier, you consider only the counties in the tri-state area and New England, i.e., those that belong to the states of New York, New Jersey, Maine, New Hampshire, Pennsylvania, Vermont, Massachusetts, Connecticut, or Rhode Island.

Your goal is to find a linear classifier that, for each county, labels it either as a Bernie win or as a Hillary win. To do this, you have access to a feature vector comprising the following features: mean income, percentage of hispanics, percentage of whites, percentage of residents with a Bachelor's degree or higher, and population density.

1. Load the data file `Hillary_vs_Bernie` in MATLAB. In `features_train`, we have given you the feature vectors for 175 counties and in `label_train`, their corresponding labels (-1 is a Bernie win and 1 is a Hillary win). As there was a wide disparity in the orders of magnitude of the original data (average income is around  $10^4$  whereas the percentages are between 0 and 1), each feature vector has already been normalized by its standard deviation. The original data can be found at <https://www.kaggle.com/benhamner/2016-us-election> (as fact checking is popular at the moment ;). Solve the problem below

$$\begin{aligned} & \min_{a,b,\eta} \|a\| + \gamma \|\eta\|_1 \\ \text{s.t. } & y_i(a^T x_i - b) \geq 1 - \eta_i \text{ for all } i = 1, \dots, m \\ & \eta_i \geq 0 \text{ for all } i = 1, \dots, m. \end{aligned}$$

to build a linear classifier for this training set for  $\gamma = 0.1, 1, 10$ . For each value of  $\gamma$ , specify the optimal  $a^*$  and  $b^*$  obtained.

2. Test the performance of your classifier using the feature vectors from 21 other counties (given in `features_test`) by comparing the labels obtained to the ones given in

`label_test`. Which  $\gamma$  gives you the highest success rate in terms of prediction? Take a look at the entries of  $a^*$  in this case – what does this suggest about the people who vote for Hillary compared to those who vote for Bernie?

**Problem 2: Norms defined by convex sets**

Define  $M_C$  as the following function of a convex set  $C$  in  $\mathbb{R}^n$ :

$$M_C(x) = \inf\{t > 0 \mid \frac{x}{t} \in C\},$$

over the domain

$$\text{dom}(M_C) = \{x \in \mathbb{R}^n \mid \frac{x}{t} \in C \text{ for some } t > 0\}.$$

1. Show that  $M_C$  is a convex function.
2. Suppose  $C$  is also compact, origin symmetric ( $x \in C$  if and only if  $-x \in C$ ), and has nonempty interior. Show that  $M_C$  is a norm over  $\mathbb{R}^n$ . What is its unit ball?
3. Show that an even degree homogeneous polynomial is convex if and only if it is quasi-convex. (Hint: use what you proved in the previous parts of this question.)

**Problem 3: Totally unimodular matrices**

Let  $A$  be an integral matrix. Show that  $A$  is totally unimodular if and only if for every integral vector  $b$ , the polyhedron  $\{x \mid x \geq 0, Ax \leq b\}$  is integral. (Hint: If  $A$  is not totally unimodular, use the inverse of a submatrix which does not have determinant  $\{0, -1, +1\}$  to construct an integer vector  $b$  that generates a non-integral vertex in the polyhedron.)

**Problem 4: True or False?** (Give a proof or a counterexample.)

- A closed convex set in  $\mathbb{R}^n$  coincides with the intersection of all halfspaces that contain it.
- Suppose  $P \in \mathbb{R}^{n \times n}$  is a matrix with nonnegative entries whose columns each sum up to one. Then, there exists  $x \in \mathbb{R}^n$  such that  $Px = x$ ,  $x \geq 0$ , and  $\sum_{i=1}^n x_i = 1$ .

**Problem 5: Radiation treatment planning (from [1])**

In radiation treatment, radiation is delivered to a patient, with the goal of killing or damaging the cells in a tumor, while carrying out minimal damage to other tissue. The radiation is

delivered in beams, each of which has a known pattern; the level of each beam can be adjusted. (In most cases multiple beams are delivered at the same time, in one ‘shot’, with the treatment organized as a sequence of ‘shots’.) We let  $b_j$  denote the level of beam  $j$ , for  $j = 1, \dots, n$ . These must satisfy  $0 \leq b_j \leq B^{\max}$ , where  $B^{\max}$  is the maximum possible beam level. The exposure area is divided into  $m$  voxels, labeled  $i = 1, \dots, m$ . The dose  $d_i$  delivered to voxel  $i$  is linear in the beam levels, i.e.,  $d_i = \sum_{j=1}^n A_{ij}b_j$ . Here  $A \in \mathbb{R}_+^{m \times n}$  is a (known) matrix that characterizes the beam patterns. We now describe a simple radiation treatment planning problem.

A (known) subset of the voxels,  $\mathcal{T} \subset \{1, \dots, m\}$ , corresponds to the tumor or target region. We require that a minimum radiation dose  $D^{\text{target}}$  be administered to each tumor voxel, i.e.,  $d_i \geq D^{\text{target}}$  for  $i \in \mathcal{T}$ . For all other voxels, we would like to have  $d_i \leq D^{\text{other}}$ , where  $D^{\text{other}}$  is a desired maximum dose for non-target voxels. This is generally not feasible, so instead we settle for minimizing the penalty

$$E = \sum_{i \notin \mathcal{T}} (d_i - D^{\text{other}})_+,$$

where  $(\cdot)_+$  denotes the nonnegative part of its argument (i.e.,  $(z)_+ = \max\{0, z\}$ ). We can interpret  $E$  as the total nontarget excess dose.

1. Show that the treatment planning problem is a linear program. The optimization variable is  $b \in \mathbb{R}^n$ ; the problem data are  $B^{\max}$ ,  $A$ ,  $\mathcal{T}$ ,  $D^{\text{target}}$ , and  $D^{\text{other}}$ .
2. Solve the problem instance with data generated by the file `treatment_planning_data.m`. Here we have split the matrix  $A$  into **Atarget**, which contains the rows corresponding to the target voxels, and **Aother**, which contains the rows corresponding to other voxels. Plot the dose histogram for the target voxels, and also for the other voxels. (You can use the MATLAB function `hist` to plot histograms.) Make a brief comment on what you see. *Remark:* The beam pattern matrix in this problem instance is randomly generated, but similar results would be obtained with realistic data.

## References

- [1] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2009. Additional Exercises. Courtesy of Stephen Boyd.