ORF 523	PROBLEM SET 4	Spring 2019, Princeton University
Instructor: A.A. Ahmadi		
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Problem 1: A nuclear program for peaceful reasons

A popular matrix norm in machine learning these days is the so-called *nuclear norm*. The nuclear norm of a matrix $A \in \mathbb{R}^{m \times n}$ is given by

$$||A||_* := \sum_{i=1}^{\min\{m,n\}} \sigma_i(A),$$

where σ_i is the i-th singular value of A. There is considerable interest in this norm partly because it serves as the convex envelope of the function rank(A) over the set $\{A \in \mathbb{R}^{m \times n} |\ ||A||_2 \leq\ 1\}$.[1] There are numerous application areas where one would like to minimize the rank of a matrix; routine examples are collaborative filtering (`http://www.netflixprize.com/`) or nonconvex quadratic programming.

1. Show that the dual norm of the spectral norm is the nuclear norm.
2. Plot the unit ball of the nuclear norm for symmetric 2×2 matrices.
3. Show that the problem of minimizing the nuclear norm of a matrix subject to arbitrary affine constraints can be cast as a semidefinite program.

Problem 2: Distance geometry

You are given a list of distances d_{ij} for $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, m\}$. You would like to know whether there are points $x_i \in \mathbb{R}^n$, for some value of n, such that

$$||x_i - x_j||_2 = d_{ij}, \forall i, j.$$

1. Show that this problem can be formulated as a semidefinite program (SDP). If this SDP answers "yes", how would you recover n and the points x_i?
2. Give an example of a set of distances that respect the triangle inequality but for which there does not exist an embedding in any dimension.

[1] What does this statement simplify to in the case where A is diagonal?

Problem 3: Stability of a pair of matrices

Recall that the spectral radius of a matrix $A \in \mathbb{R}^{n\times n}$, denoted by $\rho(A)$, is the maximum of the absolute values of its eigenvalues. We call a matrix "stable" if $\rho(A) < 1$. Let us call a pair of real $n \times n$ matrices $\{A_1, A_2\}$ stable if $\rho(\Sigma) < 1$, for any finite product Σ out of A_1 and A_2. (For example, Σ could be $A_2A_1, A_1A_2, A_1A_1A_2A_1$, and so on.)

1. Does stability of A_1 and A_2 imply stability of the pair $\{A_1, A_2\}$?

2. Prove (possibly using optimization) that the pair $\{A_1, A_2\}$ with

$$A_1 = \frac{1}{4}\begin{pmatrix} -1 & -1 \\ -4 & 0 \end{pmatrix}, A_2 = \frac{1}{4}\begin{pmatrix} 3 & 3 \\ -2 & 1 \end{pmatrix}$$

   is stable.

Problem 4: Robust linear programming

Let $c, \bar{a}_1, \ldots, \bar{a}_m \in \mathbb{R}^n$ be a set of vectors, $P_1, \ldots, P_m \in S^{n\times n}$ a set of positive definite matrices, and $b_1, \ldots, b_m \in \mathbb{R}$ a set of scalars. A *robust linear program with ellipsoidal uncertainty* is an optimization problem of the form

$$\begin{array}{ll} \min\limits_{x\in\mathbb{R}^n} & c^T x \\ \text{s.t.} & a_i^T x \le b_i, \forall a_i \in U_i, \forall i \in \{1, \ldots m\}, \end{array} \tag{1}$$

where

$$U_i = \{\bar{a}_i + P_i u \mid \|u\|_2 \le 1\}.$$

In other words, this is an LP where the rows of the constraint matrix are uncertain, but each known to be within an ellipsoid centered at a nominal vector. Show that problem (1) can be written as an SOCP.