Name:________________________________

PRINCETON UNIVERSITY

ORF 523
Midterm Exam 2, Spring 2019

APRIL 23, 2019, FROM 1:30 PM TO 2:50 PM

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PLEASE DO NOT TURN THIS PAGE. START THE EXAM ONLY WHEN YOU ARE INSTRUCTED TO DO SO.

1. You are allowed a single sheet of A4 paper, double sided, hand-written or typed.
2. No electronic devices are allowed (e.g., cell phones, calculators, laptops, etc.).
3. All questions should be answered in the booklet. Please write out and sign the following pledge on the booklet: "I pledge my honor that I have not violated the honor code during this examination."
4. You can cite results proven in lecture or on problem sets without proof.
5. Each problem has 25 points. Good luck!

Problem 1: Let $\alpha(G)$ and $\theta(G)$ respectively denote the stability number and the Shannon capacity of a graph $G(V, E)$, and $G_A \otimes G_B$ denote the strong graph product between two graphs $G_A(V_A, E_A)$ and $G_B(V_B, E_B)$. Let

$$\begin{aligned} \vartheta'(G) \ := \ \max_{X \in S^{n\times n}} \quad & \mathrm{Tr}(JX) \\ \text{s.t.} \quad & \mathrm{Tr}(X) = 1 \\ & X_{ij} = 0 \quad \text{if} \quad \{i, j\} \in E \\ & X \succeq 0 \\ & X \geq 0 \text{ (the inequlity is entrywise),} \end{aligned}$$

where J is the matrix of all ones. It is known that there are graphs for which ϑ' is strictly smaller than θ. Determine whether each of the following statements is true or false. Justify.

(a) $\alpha(G) \leq \vartheta'(G)$,

(b) $\vartheta'(G_A \otimes G_B) \leq \vartheta'(G_A) \cdot \vartheta'(G_B)$.

Problem 2: Recall that a convex QCQP is a problem of the form

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \ & x^T Q_0 x + q_0^T x + c_0 \\ \text{s.t.} \ & x^T Q_i x + q_i^T x + c_i \leq 0, \ i = 1, \ldots, m, \end{aligned}$$

where $Q_0, Q_1, \ldots, Q_m \succeq 0$. Show that this problem can be written as an SDP. (Your SDP does not need to be in standard form.)

Problem 3: Show that the largest eigenvalue of a real symmetric matrix is a convex function of its entries.

Problem 4: Consider the sets

$$E_1 = \{x \in \mathbb{R}^n \mid \ x^T P x + q^T x + r \leq 0\}, \qquad E_2 = \{x \in \mathbb{R}^n \mid \ Ax \leq b\},$$

where $A \in \mathbb{R}^{m \times n}$ and $r < 0$. Write down an SDP which is feasible if and only if $E_1 \subseteq E_2$. (Your SDP does not need to be in standard form.)