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| ORF 523 | Problem set 2 | Spring 2024, Princeton University |
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| Due on February 27, 2024, at 1:30pm EST, on Gradescope | | |

**Problem 1: True or False?**

Specify whether each of the following statements is true or false and provide either a proof or a counterexample depending on your answer. Let $S$ be a set in $\mathbb{R}^n$.

1. The convex hull of $S$ is the intersection of all convex sets that contain $S$.
2. If $S$ is closed, then the convex hull of $S$ is closed.
3. If $S$ is bounded, then the convex hull of $S$ is bounded.
4. If $S$ is compact, then the convex hull of $S$ is compact.
   (You may want to use the following fact from analysis: the image of a compact set under a continuous mapping is compact.)
5. The sum of two quasiconvex functions is quasiconvex.
6. A quadratic function $f(x) = x^T Qx + b^T x + c$ is convex if and only if it is quasiconvex.
   (You can use the fact that $f$ is convex if and only if $Q \succeq 0$ if you need to.)
7. Any closed convex set $\Omega \subseteq \mathbb{R}^n$ can be written as $\Omega = \{x \in \mathbb{R}^n \mid g(x) \leq 0\}$ for some convex function $g : \mathbb{R}^n \to \mathbb{R}$.
8. If $f : \mathbb{R}^n \to \mathbb{R}$ is convex on a convex set $S \subseteq \mathbb{R}^n$, then $f$ is continuous on $S$.
9. Suppose $P \in \mathbb{R}^{n\times n}$ is a matrix with nonnegative entries whose columns each sum up to one. Then, there exists $x \in \mathbb{R}^n$ such that $Px = x$, $x \geq 0$, and $\sum_{i=1}^n x_i = 1$.
10. A continuous function $f : \mathbb{R}^n \to \mathbb{R}$ satisfying the midpoint convexity property

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y)\ \forall x, y \in \mathbb{R}^n$$

is convex.

**Problem 2: CVX(PY) warmup / Minimum fuel optimal control**

(Boyd&Vandenberghe, Problem 4.16)

We consider a linear dynamical system with state $x(t) \in \mathbb{R}^n, t = 0, \ldots, N$, and actuator or input signal $u(t) \in \mathbb{R}$, for $t = 0, \ldots, N-1$. The dynamics of the system is given by the linear recurrence

$$x(t+1) = Ax(t) + bu(t), \quad t = 0, \ldots, N-1,$$

where $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ are given. We assume that the initial state is zero, i.e. $x(0) = 0$. The *minimum fuel optimal control problem* is to choose the inputs $u(0), \ldots, u(N-1)$ so as to minimize the total fuel consumed, which is given by

$$F = \sum_{t=0}^{N-1} f(u(t)),$$

subject to the constraint that $x(N) = x_{\text{des}}$, where $N$ is the (given) time horizon, and $x_{\text{des}} \in \mathbb{R}^n$ is the (given) desired final or target state. The function $f : \mathbb{R} \to \mathbb{R}$ is the *fuel use map* for the actuator, and gives the amount of fuel used as a function of the actuator signal amplitude. In this problem we use

$$f(a) = \begin{cases} |a| & |a| \leq 1 \\ 2|a| - 1 & |a| > 1. \end{cases}$$

This means that fuel use is proportional to the absolute value of the actuator signal, for actuator signals between $-1$ and 1; for larger actuator signals the marginal fuel efficiency is half.

(a) Formulate the minimum fuel optimal control problem as a linear program, i.e., a convex optimization problem with affine objective and constraint functions.

(b) Solve the minimum fuel optimal control problem using CVX or CVXPY for the instance with problem data

$$A = \begin{bmatrix} -1 & 0.4 & 0.8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \\ 0.3 \end{bmatrix}, \qquad x_{\text{des}} = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \qquad N = 30.$$

Plot the actuator signal $u(t)$ as a function of time $t$ using the function `stairs` (In Python you need to import `matplotlib` first). You are allowed to let CVX or CVXPY formulate the LP for you, but it's a good idea to check the answer against the LP that you formulated in the previous part.