ORF 523	PROBLEM SET 4	Spring 2024, Princeton University
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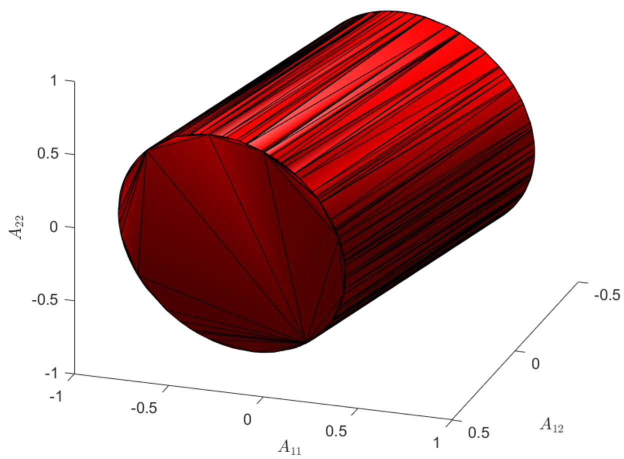
Due on April 4, 2024, at 1:30pm EST, on Gradescope		

Problem 1: A nuclear program for peaceful reasons

The *nuclear norm* of a matrix $A \in \mathbb{R}^{m \times n}$ is given by

$$\|A\|_* := \sum_{i=1}^{\min\{m,n\}} \sigma_i(A),$$

where σ_i is the i-th singular value of A. The unit ball of this norm for symmetric 2×2 matrices is plotted below.



In optimization and machine learning, there is interest in the nuclear norm partly because it serves as the convex envelope of the function $\text{rank}(A)$ over the set $\{A \in \mathbb{R}^{m \times n} | \ \|A\|_2 \leq \ 1\}$.[1] There are numerous application areas where one would like to minimize the rank of a matrix subject to affine constraints; examples include collaborative filtering or nonconvex quadratic programming.

1. Show that the dual norm of the spectral norm is the nuclear norm.

2. Show that the problem of minimizing the nuclear norm of a matrix subject to arbitrary affine constraints can be cast as a semidefinite program.

[1]What does this statement simplify to in the case where A is diagonal?

Problem 2: Distance geometry

You are given a list of distances d_{ij} for $(i,j) \in \{1,\ldots,m\} \times \{1,\ldots,m\}$. You would like to know whether there are points $x_i \in \mathbb{R}^n$, for some value of n, such that

$$||x_i - x_j||_2 = d_{ij}, \forall i, j.$$

1. Show that this problem can be formulated as that of checking whether a fixed matrix whose entries depend on d_{ij} is positive semidefinite. If this test passes, how would you recover n and the points x_i?

2. Give an example of a set of distances that respect the triangle inequality but for which there does not exist an embedding in any dimension.

Problem 3: Stability of a pair of matrices

Recall that the spectral radius of a matrix $A \in \mathbb{R}^{n\times n}$, denoted by $\rho(A)$, is the maximum of the absolute values of its eigenvalues. We call a matrix "stable" if $\rho(A) < 1$. Let us call a pair of real $n \times n$ matrices $\{A_1, A_2\}$ stable if $\rho(\Sigma) < 1$, for any finite product Σ out of A_1 and A_2. (For example, Σ could be $A_2A_1, A_1A_2, A_1A_1A_2A_1$, and so on.)

1. Does stability of A_1 and A_2 imply stability of the pair $\{A_1, A_2\}$?

2. Prove (possibly using optimization) that the pair $\{A_1, A_2\}$ with

$$A_1 = \frac{1}{4}\begin{pmatrix} -1 & -1 \\ -4 & 0 \end{pmatrix}, A_2 = \frac{1}{4}\begin{pmatrix} 3 & 3 \\ -2 & 1 \end{pmatrix}$$

   is stable.

Problem 4: SDPs with rational data but no rational feasible solution

Give an example of symmetric $n \times n$ matrices $A_1, \ldots, A_m$ with rational entries and rational numbers $b_1, \ldots, b_m$ such that the set

$$S := \{X \in S^{n\times n} | \operatorname{Tr}(A_iX) = b_i, i = 1, \ldots, m, X \succeq 0\}$$

is non-empty, but only contains matrices that have at least one irrational entry. Here, $S^{n\times n}$ denotes the set of symmetric $n \times n$ matrices with real entries and Tr stands for the trace operation. You can pick any value for n and m that you like as long as the above requirements are met.[2]

[2]This exercise shows why it is in general difficult for an SDP solver to return an exact feasible solution. By contrast, the situation for LPs is much nicer as a feasible LP with rational data always has a rational feasible solution.