

A harmonic analysis solution to the static basket arbitrage problem

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September 2, 2003

Abstract

We consider the problem of computing upper and lower bounds on the price of a European basket call option, given prices on other similar baskets. We focus here on an interpretation of this program as a generalized moment problem. Recent results by Berg & Maserick (1984), Putinar & Vasilescu (1999) and Lasserre (2001) on harmonic analysis on semigroups, the \mathbb{K} -moment problem and its applications to optimization, allow us to derive tractable necessary and sufficient conditions for the absence of static arbitrage between basket straddles, hence between basket calls and puts.

Keywords: Semidefinite Programming, Static Arbitrage, K-Moment Problem, Basket Options.

1 Introduction

We let $p \in \mathbf{R}_+^{n+m}$, $K \in \mathbf{R}^{n+m+1}$, $w_i \in \mathbf{R}^n$, $i = 0, \dots, n+m$ and we consider the problem of computing upper and lower bounds on the price of an European basket call option with strike K_0 and weight vector w_0 :

$$\begin{aligned} & \text{maximize/minimize} && p_0 := \mathbf{E}_\nu(w_0^T x - K_0)_+ \\ & \text{subject to} && \mathbf{E}_\nu(w_i^T x - K_i)_+ = p_i, \quad i = 1, \dots, n+m, \end{aligned} \quad (1)$$

with respect to all probability measures ν on the asset price vector $x \in \mathbf{R}_+^n$, consistent with the (given) set of observed prices p_i of options on other baskets.

We implicitly assume that all the options have the same maturity, and that, without loss of generality, the risk-free interest rate is zero (we compare prices in the forward market). We seek non-parametric bounds, i.e., we do not assume any specific model for the underlying asset prices, our only assumption is the absence of a static arbitrage today (i.e. the absence of an arbitrage that only requires trading today and at maturity).

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Here, we interpret (1) as a *generalized moment problem*. This approach was successfully used in [BP02] to get tractable bounds in dimension one and to show the NP-hardness of the multivariate problem (1). NP-hardness means that we have no chance of finding a direct and efficient method for detecting all arbitrage opportunities, here instead we look for a sequence of successively tighter price bounds. This means that outlandish arbitrage opportunities can be detected at little numerical cost while detecting finer price discrepancies has a higher theoretical complexity.

Recent results on multivariate moment problems (see [Sch91], [PV99] or [CF00]), semidefinite programming (see [NN94], [VB96] and [Nes00]) and harmonic analysis on semigroups (see [BCR84] and [Rom03]) allow us to derive static arbitrage price bounds on a set of products linked by a semigroup structure. The resulting constraints can be formulated as successively tighter linear matrix inequalities, hence we can compute increasingly sharp bounds on the solution to problem (1) as solutions of increasingly large semidefinite programs (linear programs on the cone of positive semidefinite matrices). Semidefinite programming has been the object of intensive research since the seminal work of [NN94] and several numerical packages (see for example SEDUMI by [Stu99]) are now available to solve these problems very efficiently.

The core of our argument is to substitute to the classical duality between the cones of probability measures and positive portfolios, the conic duality between positive definite functions on one hand and sums of squares on the other. These last two cones have the advantage of being numerically tractable and lead to exploitable formulations of the static portfolio super/sub-replication problems.

A lot of work has been focused on arbitrage bounds in a dynamic setting, see [EKQ91], [EKQ95], [ALP95], and [KS98], among others. Work on the unidimensional static problem dates back at least to [BL78] (see also [LL00]), both using the positivity of butterfly spread prices to preclude arbitrage. [BP02] studied these bounds together with second order moment constraints and proved the NP-Hardness of the multivariate problem (1). Finally, in a previous paper [dEG03], we focused on the interpretations of problem (1) as an *integral transform inversion problem* or a *linear semi-infinite program*, i.e. a linear program with a finite number of linear constraints on an infinite dimensional variable, and used the related theories to compute closed-form solutions for some particular cases and a linear programming relaxation for the general case.

The paper is organized as follows. In section two, we describe the static market structure and start with a brief introduction on harmonic analysis on semigroups. Based on these results, we then derive necessary and sufficient conditions for the absence of arbitrage in the static market, formulated as semidefinite programs. Finally, in section three, we describe the conic duality between positive definite functions and sums of squares and use it to show how a super/sub-replicating portfolio can be constructed from the solution to the programs of the preceding section.

2 Static arbitrage constraints

2.1 Market structure

We work in a one period framework and suppose that the market is composed of cash and n underlying assets x_i for $i = 1, \dots, n$ with $x \in \mathbf{R}_+^n$. We suppose that the forward prices of the assets are known and given by p_i , for $i = 1, \dots, n$, hence w_i is the Euclidean basis and $K_i = 0$ for $i = 1, \dots, n$. In addition to these basic products, there are $m + 1$ basket *straddles* on the assets x , with payoff given by $|w_{n+i}^T x - K_{n+i}|$, $i = 1, \dots, m$. Because a straddle is obtained as the sum of a call and a put, we get the market price of straddles from those of basket calls and forward contracts by call-put parity.

We will note these payoff functions e_i , for $i = 0, \dots, m+n$, with $e_i(x) = x_i$ for $i = 1, \dots, n$ and $e_{(n+j)}(x) = |w_j^T x - K_j|$ for $j = 0, \dots, m$. In what follows, we will focus on the Abelian (commutative) semigroup (\mathbb{S}, \cdot) generated by the payoffs $e_i(x)$ for $i = 0, \dots, m+n$, the cash $1_{\mathbb{S}}$ and their products.

In this one period setting, we will look for conditions that guarantee the absence of *static arbitrages*, i.e. arbitrage opportunities that only involve trading today and at maturity, assuming that there are no transaction costs.

2.2 Harmonic analysis on semigroups

We start by a brief introduction on harmonic analysis on semigroups, for a complete treatment see [BCR84] and the references therein. Unless otherwise specified, all measures are supposed to be positive.

Definition 1 A function $\rho : \mathbb{S} \rightarrow \mathbf{R}$ is called a *semicharacter* iff it satisfies $\rho(st) = \rho(s)\rho(t)$ for all $s, t \in \mathbb{S}$ and $\rho(1_{\mathbb{S}}) = 1$.

In [BCR84] an involution operation is defined on the semigroup (\mathbb{S}, \cdot) , here and in the rest of the paper we suppose that involution to be the identity, which means in particular that we take all semicharacters to be real valued. The dual semigroup of \mathbb{S} , i.e. the set of semicharacters on \mathbb{S} is called \mathbb{S}^* . In this context, we call a function $f : \mathbb{S} \rightarrow \mathbf{R}$ a *moment function* on \mathbb{S} iff $f(1_{\mathbb{S}}) = 1$ and f can be represented as:

$$f(s) = \int_{\mathbb{S}^*} \rho(s) d\nu(\rho), \quad \text{for all } s \in \mathbb{S}, \quad (2)$$

where ν is a Radon measure on \mathbb{S}^* .

When \mathbb{S} is the semigroup defined in (2.1) as an enlargement of the semigroup of monomials on \mathbf{R}^n , its dual \mathbb{S}^* is the set of applications $\rho_x : \mathbb{S} \rightarrow \mathbf{R}$ such that $\rho_x(s) = s(x)$ for all $s \in \mathbb{S}$ and all $x \in \mathbf{R}^n$. The measure ν is then assimilated to a probability measure on \mathbf{R}^n and the representation above becomes:

$$f(s) = \mathbf{E}_{\nu} [s(x)], \quad \text{for all } s \in \mathbb{S}. \quad (3)$$

Our objective below is to find tractable conditions for a set of prices p_0, \dots, p_{n+m} to be represented as $\mathbf{E}_{\nu} [|w_i^T x - K_i|] = p_i$ for $i = 0, \dots, n+m$ and some positive measure ν .

2.3 The compact case

In this section we assume the asset distribution has a compact support K . We treat the compact case independently as it is rather simple yet captures many of the key features of the general result. We begin by a few definitions along the lines of [BM84] and [BCR84]. An *absolute value* on \mathbb{S} is a function $|\cdot| : \mathbb{S} \rightarrow \mathbf{R}_+$ satisfying

$$|s^2| \leq |s|^2, \quad \text{for all } s \in \mathbb{S}$$

and

$$|1_{\mathbb{S}}| \geq 1.$$

A function $f : \mathbb{S} \rightarrow \mathbf{R}$ is said to be bounded with respect to an absolute value $|\cdot|$ iff there exists some $M > 0$ such that

$$|f(s)| \leq M|s|, \quad \text{for all } s \in \mathbb{S}.$$

Furthermore, f is called *exponentially bounded* iff f is bounded with respect to some absolute value. Remark that if the measure ν in (2) has its support contained in the compact K then the moment function $f(s) = \int_{\mathbb{S}^*} \rho(s) d\nu(\rho)$ is bounded with respect to the following absolute value:

$$|s|_K = \sup_{\rho \in K} \rho(s)$$

for $s \in \mathbb{S}$.

Definition 2 *A function $f : \mathbb{S} \rightarrow \mathbf{R}$ is called positive semidefinite iff for all finite families $\{s_i\}$ of elements of \mathbb{S} , the matrix with coefficients $f(s_i s_j)$ is positive semidefinite.*

We remark that moment functions are necessarily positive semidefinite. Necessary and sufficient conditions for the existence of a measure ν in (3) were derived in [HS90], they were however numerically intractable. Here, based on the results in [BCR84], [PV99] and [Rom03], we look for exploitable conditions for representation (3) to hold.

Let α be an absolute value, the central result in [BCR84, Th. 2.6] states that the set of α -bounded positive semidefinite functions $f : \mathbb{S} \rightarrow \mathbf{R}$ such that $f(1_{\mathbb{S}}) = 1$ is a Bauer simplex whose extreme points are given by the set of α -bounded semicharacters. Hence a function f is positive semidefinite and exponentially bounded if and only if it can be represented as $f(s) = \int_{\mathbb{S}^*} \rho d\nu(\rho)$ with the support of ν included in some compact subset of \mathbb{S}^* .

Based on these results, we derive below a set of tractable necessary and sufficient conditions allowing a function f to be represented as in (3). For s, u in \mathbb{S} , we note E_s the shift operator such that for $f : \mathbb{S} \rightarrow \mathbf{R}$, we have $E_s(f(u)) = f(su)$ and we let \mathcal{E} be the commutative algebra generated by the shift operators on \mathbb{S} . Finally, we let $\beta = \sup_{x \in K} \{\sum_{i=0}^{n+m} e_i(x)\}$.

Theorem 3 *Suppose the asset distribution has compact support K and \mathbb{S} is the payoff semigroup defined in (2.1), with β is defined as above. A function $f(s) : \mathbb{S} \rightarrow \mathbf{R}$ can be represented as*

$$f(s) = \mathbf{E}_{\nu}[s(x)], \quad \text{for all } s \in \mathbb{S}, \tag{4}$$

for some measure ν on K , and satisfies the price constraints in (1) if and only if:

- (i) f is positive semidefinite,
- (ii) $E_{e_i} f$ is positive semidefinite for $i = 0, \dots, n + m$,
- (iii) $(\beta I - \sum_{i=0}^{n+m} E_{e_i}) f$ is positive semidefinite,
- (iv) $f(e_i) = p_i$ for $i = 1, \dots, n + m$.

Furthermore, for each function f satisfying conditions (i) to (iv), the measure ν in representation (4) is unique.

Proof. The family of shift operators $\tau = \{\{E_{e_i}\}_{i=0, \dots, n+m}, (\beta I - \sum_{i=0}^{n+m} E_{e_i})\} \subset \mathcal{E}$ is such that $I - T \in \text{span}^+ \tau$ for each $T \in \tau$ and $\text{span } \tau = \mathcal{E}$, hence τ is linearly admissible in the sense of [BM84, Corollary 2.5] or [Mas77], which states that (ii) and (iii) are equivalent to f being τ -positive. Then, [Mas77, Th. 2.1] means that f is τ -positive if and only if there is a measure ν such that $f(s) = \int_{\mathbb{S}^*} \rho(s) d\nu(\rho)$, whose support is a compact subset of the τ -positive semicharacters. This means in particular that for a semicharacter $\rho_x \in \text{supp}(\nu)$ we must have $\rho_x(e_i) \geq 0$, for $i = 1, \dots, n$ hence $x \geq 0$. The set of τ -positive semicharacters is then included in the nonnegative orthant and includes both the simplex $\{x \geq 0 : \|x\|_1 \leq \beta\}$ and \mathbf{K} , hence f being τ -positive is equivalent to f admitting a representation of the form $f(s) = \mathbf{E}_\nu [s(x)]$, for all $s \in \mathbb{S}$ with ν having a compact support $K \subset \mathbf{R}_+^n$. ■

2.4 The unbounded case

The conditions derived in the last part do not describe all possible arbitrage free prices as they cannot account for unbounded asset distributions. Here, we use results from [PV99] and [Rom03] to derive intrinsic characterizations of viable multivariate straddle prices.

We note $\mathcal{A}(\mathbb{S})$ the \mathbf{R} -algebra generated by the functions $\chi_s : \mathbb{S}^* \rightarrow \mathbf{R}$ such that $\chi_s(\rho) = \rho(s)$ for all $s \in \mathbb{S}$. By construction, $\chi_s(\rho) = E_s \rho(1_{\mathbb{S}})$, and for a polynomial $p \in \mathcal{A}(\mathbb{S})$ with $p = \sum_k q_k \chi_{g_k}$ and for $\rho \in \mathbb{S}^*$ we have $p\rho(s) = \sum_k q_k \rho(sg_k)$ for all $s \in \mathbb{S}$. When \mathbb{S} is the payoff semigroup defined in (2.1), we naturally have $\chi_s(\rho_x) = s(x)$, for all $x \in \mathbf{R}^n$, $s \in \mathbb{S}$ and $\rho \in \mathbb{S}^*$.

We now note $\mathcal{A}_\theta(\mathbb{S})$ the \mathbf{R} -algebra generated by $\mathcal{A}(\mathbb{S})$ and θ where

$$\theta(\rho) = \left(1 + \sum_{i=0}^{m+n} \chi_{e_i^2}(\rho) \right)^{-1}, \quad \text{for all } \rho \in \mathbb{S}^*, \quad (5)$$

we also note $\mathcal{A}(\mathbb{S}, y)$ the algebra generated by $\mathcal{A}(\mathbb{S})$ and $\mathbf{R}[y]$. We first simplify the equality constraints on 2^n variables in [Rom03, Th. A] to recover an additive formulation as in [PV99]. We begin by proving the following lemma.

Lemma 4 *The kernel of the algebra homomorphism Φ :*

$$\begin{aligned} \mathcal{A}(\mathbb{S}, y) &\rightarrow \mathcal{A}_\theta(\mathbb{S}) \\ p(\rho, y) &\mapsto \Phi p = p(\rho, \theta(\rho)) \end{aligned} \quad (6)$$

is the ideal generated by $\sigma \in \mathcal{A}(\mathbb{S}, y)$ such that $\sigma(\rho, y) = y(1 + \sum_{i=1}^{m+n+1} \chi_{e_i^2}(\rho)) - 1$.

Proof. We adapt the proof of [PV99, lemma 2.3] and let $p \in \mathcal{A}(\mathbb{S}, y)$ be such that $p(\rho, \theta(\rho)) = 0$, we write $p(\rho, y) = \sum_k q_k(\rho)y^k$ with $q_k \in \mathcal{A}(\mathbb{S})$. We have:

$$\begin{aligned} p(\rho, y) &= p(\rho, y) - p(\rho, \theta(\rho)) = \sum_{k>0} q_k(\rho)(y^k - (\theta(\rho))^k) \\ &= (y - (\theta(\rho)))l(\rho, y, (\theta(\rho))), \end{aligned}$$

where l is a polynomial. Let $\kappa = \max\{k : q_k \neq 0\}$ and

$$\tau(\rho) = \left(1 + \sum_{i=1}^{m+n+1} \chi_{e_i^2}(\rho) \right)^\kappa, \quad \text{for all } \rho \in \mathbb{S}^*,$$

we then have

$$\tau(\rho)p(\rho, y) = \sigma(\rho, y)r(\rho, y), \tag{7}$$

with $r(\rho, y) \in \mathcal{A}(\mathbb{S}, y)$. The case $\kappa = 0$ is trivial hence we can assume $\kappa \neq 0$. Using the fact that the polynomials $\tau(z)$ and $\sigma(z)$ have no common zeroes in $\mathbf{C}^{m+n+2}[z]$, Hilbert's *Nullstellensatz* (see [BCR98] for example) states that there must be $\tilde{\tau}, \tilde{\sigma} \in \mathbf{C}^{m+n+2}[z]$ such that

$$\tau\tilde{\tau} + \sigma\tilde{\sigma} = 1.$$

Multiplying this last identity by p yields, together with (7):

$$p = \sigma(r\tilde{\tau} + p\tilde{\sigma})$$

hence the desired result. ■

The next proposition is adapted from the dimensional extension method in [PV99, Th. 2.5] and [Rom03, Th. 4], to replace the exponential number of equality constraints in [Rom03, Th. A] with an additive formulation as in [PV99]. The function $\theta(\rho)$ is defined as in (5) and $\mathcal{A}_\theta(\mathbb{S})$ is the \mathbf{R} -algebra generated by $\mathcal{A}(\mathbb{S})$ and θ .

Proposition 5 *With \mathbb{S} being the payoff semigroup defined in (2.1), let Λ be a positive semidefinite linear form on $\mathcal{A}_\theta(\mathbb{S})$ such that $\Lambda(x_i r^2) \geq 0$ for all $r \in \mathcal{A}_\theta(\mathbb{S})$ and $i = 1, \dots, n$, then Λ has a unique representing measure ν with support in \mathbf{R}_+^n and $\mathcal{A}_\theta(\mathbb{S})$ is dense in $L^2(\nu)$.*

Proof. We recall that the linear form Λ is positive semidefinite iff $\Lambda(r^2) \geq 0$, for all $r \in \mathcal{A}_\theta(\mathbb{S})$. As in [Rom03], we define a bilinear form on $r \in \mathcal{A}_\theta(\mathbb{S})$ by:

$$\langle r_1, r_2 \rangle := \Lambda(r_1 r_2), \quad \text{for all } r_1, r_2 \in \mathcal{A}_\theta(\mathbb{S})$$

We let \mathcal{N} be the set $\{r \in \mathcal{A}_\theta(\mathbb{S}) : \Lambda(r^2) = 0\}$. The bilinear form above then defines a scalar product on $\mathcal{A}_\theta(\mathbb{S})/\mathcal{N}$, and we note \mathcal{H} the completion of this space. We define in \mathcal{H} the operators:

$$T_i(r + \mathcal{N}) = \chi_{e_i} r + \mathcal{N}, \quad \text{for all } r \in \mathcal{A}_\theta(\mathbb{S})/\mathcal{N} \text{ and } i = 0, \dots, n+m,$$

which are symmetric and densely defined in \mathcal{H} . We also define the operator $(D(B), B)$ by:

$$D(B) = \mathcal{A}_\theta(\mathbb{S})/\mathcal{N} \text{ and } B = \sum_{i=0}^{m+n} T_i^2.$$

The operator B is positive as a sum of squares of operators and, by construction, the domain $D(B)$ is dense in \mathcal{H} and invariant by B . Let $\tau = \sum_{i=0}^{m+n} \chi_{e_i^2}(\rho)$ and $r \in \mathcal{A}_\theta(\mathbb{S})/\mathcal{N}$, then $u = r\theta$ is such that $(1 + \tau)u = r$, hence the operator $I + B$ is bijective on $D(B)$. This means that B satisfies the hypothesis of [PV99, Lemma 2.2] and is essentially self-adjoint. [Rom03, Prop. 1] then implies that the operators T_i for $i = 0, \dots, n + m$ are essentially normal and that their canonical closures commute, meaning that there exists a common spectral measure H for the operators \bar{T}_i for $i = 0, \dots, n + m$. With $T = (T_i)_{i=0, \dots, n+m}$ and $r \in \mathcal{A}_\theta(\mathbb{S})/\mathcal{N}$, we define the operator $r(T)$ by:

$$\begin{aligned} \mathcal{A}_\theta(\mathbb{S})/\mathcal{N} &\rightarrow \mathcal{A}_\theta(\mathbb{S})/\mathcal{N} \\ w + \mathcal{N} &\mapsto r(T)(w + \mathcal{N}) = rw + \mathcal{N}. \end{aligned} \quad (8)$$

With $\gamma(x) = (\sum_{i=0}^{n+m} x_i^2)^{-1}$, there is an element q of $\mathbf{R}_\gamma[x]$, the \mathbf{R} -algebra generated by $\mathbf{R}[x]$ and $\gamma(x)$ such that $r(\rho) = q((\chi_{e_i})_{i=0, \dots, n+m}(\rho), \theta(\rho))$ for all $\rho \in \mathbb{S}^*$. We then have:

$$\Lambda(r) = \langle r(T)1, 1 \rangle = \langle q(\bar{T})1, 1 \rangle = \int_{\mathbf{R}^{n+m+1}} q(x) dH_{1+\mathcal{N}, 1+\mathcal{N}}(x),$$

The homomorphism f :

$$\begin{aligned} \mathbf{R}_\gamma[x] &\rightarrow \mathcal{A}_\theta(\mathbb{S}) \\ p(x) &\mapsto f(p) = p((\chi_{e_i}(\rho))_{i=0, \dots, n+m}, \theta(\rho)) \end{aligned} \quad (9)$$

satisfies the hypothesis of [Rom03, Lemma 2] hence there is a (positive) Radon measure ν on such that:

$$\Lambda(r) = \int_{\mathbb{S}^*} r(\rho) d\nu(\rho),$$

which, if \mathbb{S} is defined as in section (2.1), is also:

$$\Lambda(r) = \int_{\mathbf{R}^n} r(x) d\nu(x).$$

Uniqueness and density follow from the argument in [Rom03]. Now, because the operators T_i for $i = 1, \dots, n$ are essentially self-adjoint with $\Lambda(x_i r^2) \geq 0$ for $r \in \mathcal{A}_\theta(\mathbb{S})$ and $i = 1, \dots, n$, we know that the T_i are positive for all i . The spectral measure F_i of T_i is given by $F_i(X) = H(\bar{T}_i^{-1}(X))$ for all Borel sets $X \subset \mathbf{R}$ and F_i must be concentrated in \mathbf{R}_+ for all $i = 1, \dots, n$ hence the spectral measure H of \bar{T} is concentrated in \mathbf{R}_+^n and so is the representing measure ν . ■

We can now formulate a general moment theorem that describes all the price systems that admit a representation as in (3).

Theorem 6 Let \mathbb{S} be defined as in (2.1). A sequence $f(s) : \mathbb{S} \rightarrow \mathbf{R}$ is a moment sequence and can be represented as in (3):

$$f(s) = \mathbf{E}_\nu[s(x)], \quad \text{for all } s \in \mathbb{S},$$

for some measure ν with support in \mathbf{R}_+^n , if and only if there is a sequence $p(s, k) : (\mathbb{S}, \mathbf{N}) \rightarrow \mathbf{R}$ such that:

(i) $p(s, 0) = f(s)$ for all $s \in \mathbb{S}$,

(ii) $p(s, k)$ is positive semidefinite on (\mathbb{S}, \mathbf{N}) ,

(iii) $p(e_i s, k)$ is positive semidefinite on (\mathbb{S}, \mathbf{N}) for $i = 1, \dots, n$,

(iv) $p(s, k) = p(s, k+1) - \sum_{i=0}^{n+m} p(e_i^2 s, k+1)$ for all $(s, k) \in (\mathbb{S}, \mathbf{N})$.

Furthermore, the representing measure for sequence f is unique if and only if the sequence p is unique.

Proof. First we show that conditions (i)-(iv) are necessary. With \mathbb{S} , the payoff semigroup defined in (2.1), we recall that \mathbb{S}^* can be identified with \mathbf{R}_+^n , hence $\chi_s(\rho_x) = s(x)$, for all $x \in \mathbf{R}_+^n$, $s \in \mathbb{S}$ and $\rho \in \mathbb{S}^*$. Suppose that f can be represented as:

$$f(s) = \int_{\mathbf{R}_+^n} s(x) d\nu(x), \quad \text{for all } s \in \mathbb{S},$$

we let

$$p(s, k) = \int_{\mathbf{R}_+^n} s(x) \left(1 + \sum_{i=0}^{m+n} e_i^2(x) \right)^{-k} d\nu(x), \quad \text{for all } (s, k) \in (\mathbb{S}, \mathbf{N}),$$

which satisfies (i) and (iv) by construction, $p(s, k)$ is then a moment sequence on the product semigroup $((\mathbb{S}, \cdot) \times (\mathbf{N}, +))$ and as such must be positive semidefinite, hence condition (ii). Then, because for $i = 1, \dots, n$ we have

$$p(e_i s, k) = \int_{\mathbf{R}_+^n} s(x) \left(1 + \sum_{i=0}^{m+n} e_i^2(x) \right)^{-k} e_i(x) d\nu(x), \quad \text{for all } (s, k) \in (\mathbb{S}, \mathbf{N}),$$

we know that $p(e_i s, k)$ is a moment sequence for the measure $e_i(x) d\nu$, hence condition (iii).

Conversely, let's assume that we are given a sequence $p(s, k)$ satisfying (i)-(iv). We let $\mathcal{A}_\theta(\mathbb{S})$ and $\mathcal{A}(\mathbb{S}, y)$ be the \mathbf{R} -algebras described at the beginning of the section. We define a linear function Λ on $\mathcal{A}(\mathbb{S}, y)$ by:

$$L \left(\sum_{j,k} a_j \chi_{s_j} y^k \right) = \sum_{j,k} a_j p(s_j, k)$$

and as in lemma 4, we can define the following algebra homomorphism Φ :

$$\begin{aligned} \mathcal{A}(\mathbb{S}, y) &\rightarrow \mathcal{A}_\theta(\mathbb{S}) \\ p(\rho, y) &\mapsto \Phi p = p(\rho, \theta(\rho)) \end{aligned} \tag{10}$$

whose kernel \mathcal{N} has been computed in lemma 4, and $\mathcal{A}_\theta(\mathbb{S})$ is isomorphic to the quotient $\mathcal{A}(\mathbb{S}, y)/\mathcal{N}$. Condition (iv) implies that $L(\mathcal{N}) = 0$ and we can then define a linear form Λ on $\mathcal{A}_\theta(\mathbb{S})$ by:

$$\Lambda(r) = L(q), \quad \text{where } r(\rho) = q(\rho, \theta(\rho)), \quad \text{for all } \rho \in \mathbb{S}^*,$$

with $r \in \mathcal{A}_\theta(\mathbb{S})$ and $q \in \mathcal{A}(\mathbb{S}, y)$. Because of (i)-(iv), the form Λ satisfies the hypothesis of proposition 5 and has a unique representing measure ν . ■

3 Price bounds and static hedging

In this section, we show how the duality between the existence of a pricing measure and that of a replicating portfolio transposes into the moment framework described in the previous section. In particular, we detail how an optimal static super/sub-replicating portfolio can be constructed using the solution to the dual of to the moment problem in (1). In particular, in a result that is consistent with the dynamic framework (see [ALP95]), the replicating portfolio only involves options in the data set and no other option is needed to "complete the grid".

3.1 Price bounds via semidefinite programming

Here, we show how one can compute bounds on the solution of problem (1) using a subset of the moment conditions imposed by theorem 6. These conditions cast (3) as a semidefinite program (see [NN94] or [VB96]), which can then be solved efficiently using solvers such as SEDUMI by [Stu99].

3.1.1 Asset distributions with compact support

As before, we note $\mathcal{A}(\mathbb{S})$ the \mathbf{R} -algebra generated by the functions $\chi_s : \mathbb{S}^* \rightarrow \mathbf{R}$ such that $\chi_s(\rho) = \rho(s)$ for all $s \in \mathbb{S}$ and $\rho \in \mathbb{S}^*$. For a polynomial $p \in \mathcal{A}(\mathbb{S})$ with $p = \sum_i q_i \chi_{g_i}$ where $g_i \in \mathbb{S}$, and for $s \in \mathbb{S}$ we set

$$p\rho(s) = \sum_i q_i \rho(sg_i).$$

With \mathbb{S} the payoff semigroup defined in (2.1), we recall that \mathbb{S}^* can be identified with \mathbf{R}^n , hence $\chi_s(\rho_x) = s(x)$, for all $x \in \mathbf{R}^n$, $s \in \mathbb{S}$ and $\rho_x \in \mathbb{S}^*$. This means that $p \in \mathcal{A}(\mathbb{S})$ can be rewritten

$$p(x) = \sum_i q_i s(x) g_i(x), \quad \text{for all } x \in \mathbf{R}_+^n.$$

We now recall the construction of moment matrices as in [CF00] and [Las01]. We adopt the following multiindex notation for monomials in $\mathcal{A}(\mathbb{S})$:

$$e^\alpha(x) := e_0^{\alpha_0}(x)e_1^{\alpha_1}(x) \cdots e_{m+n}^{\alpha_{m+n}}(x),$$

and we let

$$y_e = (1, e_0, \dots, e_{m+n}, e_0^2, e_0e_1, \dots, e_0^d, \dots, e_{m+n}^d) \quad (11)$$

be the vector of all monomials in $\mathcal{A}(\mathbb{S})$, up to degree d , listed in graded lexicographic order. We note $s(d)$ the size of the vector y_e . Let $y \in \mathbf{R}^{s(2d)}$ be the vector of moments (indexed as in y_e) of some probability measure ν with support in \mathbf{R}_+^n , we note $M_d(y) \in \mathbf{R}^{s(d) \times s(d)}$, the symmetric matrix:

$$M_d(y)_{i,j} = \int_{\mathbf{R}_+^n} (y_e)_i(x) (y_e)_j(x) d\nu(x), \quad \text{for } i, j = 1, \dots, s(d)$$

In the rest of the paper, we will always implicitly assume that $y_1 = 1$. With $\beta(i)$ the exponent of the monomial $(y_\chi)_i$ and conversely, $i(\beta)$ the index of the monomial e^β in y_e . We notice that for a given moment vector $y \in \mathbf{R}^{s(d)}$ ordered as in (14), the first row and columns of the matrix $M_d(y)$ are then equal to y . The rest of the matrix is then constructed according to:

$$M_d(y)_{i,j} = y_{i(\alpha+\beta)} \text{ if } M_d(y)_{i,1} = y_{i(\alpha)} \text{ and } M_d(y)_{1,j} = y_{i(\beta)}.$$

Similarly, let $g \in \mathcal{A}(\mathbb{S})$, we derive the moment matrix for the measure $g(x)d\nu$ on \mathbf{R}_+^n (called the localizing matrix in [CF00]), noted $M_d(gy) \in \mathbf{S}^{s(d)}$, from the matrix of moments $M_d(y)$ by:

$$M_d(gy)_{i,j} = \int_{\mathbf{R}_+^n} (y_e)_i(x) (y_e)_j(x) g(x) d\nu(x)$$

for $i, j = 1, \dots, s(d)$. The coefficients of the matrix $M_m(gy)$ are then given by:

$$M_d(gy)_{i,j} = \sum_{\alpha} g_{\alpha} y_{i(\beta(i)+\beta(j)+\alpha)} \quad (12)$$

We can then form a semidefinite program to compute a lower bound on the optimal solution to (1) using a subset of the moment constraints in theorem 4, taking only monomials and moments in y up to a certain degree.

Corollary 7 *Let N be a positive integer and $y \in \mathbf{R}^{s(2N)}$, a lower bound on the optimal value of:*

$$\begin{aligned} & \text{minimize } p_0 := \mathbf{E}_{\nu}[e_0(x)] \\ & \text{subject to } \mathbf{E}_{\nu}[e_i(x)] = p_i, \quad i = 1, \dots, n+m, \end{aligned}$$

can be computed as the solution of the following semidefinite program:

$$\begin{aligned} & \text{minimize } y_2 \\ & \text{subject to } M_N(y) \succeq 0 \\ & \quad M_N(e_j y) \succeq 0, \quad \text{for } j = 1, \dots, n, \\ & \quad M_N\left(\left(\beta - \sum_{k=0}^{n+m} e_k\right)y\right) \succeq 0 \\ & \quad y_{(j+2)} = p_j, \quad \text{for } j = 1, \dots, n+m \text{ and } s \in \mathbb{S} \end{aligned} \quad (13)$$

where s is such that $i(s) \leq s(2N)$. The optimal value of (13) converges to the optimal value of the original program as $N \rightarrow \infty$.

3.1.2 Unbounded distributions

Here we work on the product semigroup $(\mathbb{S}, \cdot) \times (\mathbf{N}, +)$. Its dual is the set of functions $\rho_x : (\mathbb{S}, \mathbf{N}) \rightarrow \mathbf{R}$ such that $\rho_x((s, k)) = s(x)x^k$ for all $s \in \mathbb{S}$, $k \in \mathbf{N}$ and $x \in \mathbf{R}^n$. As before, we note $\mathcal{A}(\mathbb{S}, \mathbf{N})$ the \mathbf{R} -algebra generated by the functions $\chi_s : (\mathbb{S}, \mathbf{N})^* \rightarrow \mathbf{R}$ such that $\chi_{(s,k)}(\rho) = \rho((s, k))$ for all $s \in \mathbb{S}$ and $\rho \in (\mathbb{S}, \mathbf{N})^*$. With \mathbb{S} , the payoff semigroup defined in (2.1), here $(\mathbb{S}, \mathbf{N})^*$ can again be identified with \mathbf{R}^n , hence $\chi_{(s,k)}(\rho_x) = s(x)x^k$, for all $x \in \mathbf{R}^n$, $(s, k) \in (\mathbb{S}, \mathbf{N})$ and $\rho_x \in (\mathbb{S}, \mathbf{N})^*$. By construction, we have

$$(\chi_{(s,k)})^2 = \chi_{(s^2, 2k)}, \quad \text{for all } (s, k) \in (\mathbb{S}, \mathbf{N}),$$

and for a polynomial $p \in \mathcal{A}(\mathbb{S}, \mathbf{N})$ with $p = \sum_i q_i \chi_{g_i} x^{k_i}$ where $(g_i, k_i) \in (\mathbb{S}, \mathbf{N})$, and for $(s, l) \in (\mathbb{S}, \mathbf{N})$ we set

$$p((s, l))(x) = \sum_i q_i s(x) g_i(x) x^{k_i+l},$$

for all $x \in \mathbf{R}^n$. We adopt here the multiindex notation for monomials in $\mathcal{A}(\mathbb{S}, \mathbf{N})$:

$$e^\alpha := (e_0, 0)^{\alpha_0} (e_1, 0)^{\alpha_1} \cdots (e_{m+n}, 0)^{\alpha_{m+n}} (1, 1)^{\alpha_{m+n+1}}.$$

We then let

$$y_e = (1, (e_0, 0), \dots, (e_{m+n}, 0), (1, 1), (e_0, 0)^2, (e_0, 0)(e_1, 0), \dots, (e_0, 0)^d, \dots, (1, 1)^d) \quad (14)$$

be the vector of all monomials in $\mathcal{A}(\mathbb{S}, \mathbf{N})$, up to degree d , listed in graded lexicographic order. We note $s(d)$ the size of the vector y_e . The matrices $M_d(y)$ and $M_d(gy)$ are defined as in the compact case above.

We can again form a semidefinite program, this time using a subset of the moment constraints in theorem 6, taking only moments up to a certain degree.

Corollary 8 *Let N be a positive integer and $y \in \mathbf{R}^{s(2N)}$, a lower bound on the optimal value of:*

$$\begin{aligned} & \text{minimize } p_0 := \mathbf{E}_\nu[e_0(x)] \\ & \text{subject to } \mathbf{E}_\nu[e_i(x)] = p_i, \quad i = 1, \dots, n+m, \end{aligned}$$

can be computed as the solution of the following semidefinite program:

$$\begin{aligned} & \text{minimize } y_2 \\ & \text{subject to } M_N(y) \succeq 0 \\ & \quad M_N((e_j, 0)y) \succeq 0, \quad \text{for } j = 1, \dots, n, \\ & \quad y_{i(s,k)} = y_{i(s,k+1)} - \sum_{i=0}^{n+m} y_{i(e_i^2 s, k+1)} \\ & \quad y_{(j+1)} = p_j, \quad \text{for } j = 1, \dots, n+m \text{ and } (s, k) \in (\mathbb{S}, \mathbf{N}) \end{aligned} \quad (15)$$

where (s, k) are taken such that $i(s, k) \leq s(2N)$. The optimal value of (15) converges to the optimal value of the original program as $N \rightarrow \infty$.

3.2 Static hedging portfolios and sums of squares

We let here $\Sigma \subset \mathcal{A}(\mathbb{S})$ be the set of polynomials that are sums of squares of polynomials in $\mathcal{A}(\mathbb{S})$, and \mathcal{P} the set of positive semidefinite sequences on \mathbb{S} . The central argument of this paper is to replace the conic duality between probability measures and positive portfolios:

$$p(x) \geq 0 \Leftrightarrow \int p(x) d\nu \geq 0, \quad \text{for all measures } \nu,$$

by the conic duality between positive semidefinite sequences \mathcal{P} and sums of squares polynomials Σ :

$$\langle f, p \rangle \geq 0 \text{ for all } p \in \Sigma \text{ iff } f \in \mathcal{P}.$$

for $p \in \mathcal{A}(\mathbb{S})$ with $p = \sum_i q_i \chi_{s_i}$ and $f : \mathbb{S} \rightarrow \mathbf{R}$ having defined $\langle f, p \rangle = \sum_i q_i f(s_i)$. The previous section used positive semidefinite sequences to characterize viable price sets, in this section, we use sums of squares polynomials to characterize super/sub-replicating portfolios.

From the initial price problem (1) written in terms of straddles:

$$\begin{aligned} & \text{minimize} && p_0 := \int_{\mathbf{R}_+^n} e_0(x) d\nu(x) \\ & \text{subject to} && \int_{\mathbf{R}_+^n} e_i(x) d\nu(x) = p_i, \quad i = 1, \dots, n+m, \\ & && \int_{\mathbf{R}_+^n} d\nu(x) = 1, \end{aligned} \tag{16}$$

in the variable ν , a positive measure on \mathbf{R}_+^n . We can form the Lagrangian:

$$L(\nu, \lambda) = \lambda_{n+m+1} + \sum_{i=1}^{n+m} \lambda_i p_i + \int_{\mathbf{R}_+^n} \left(e_0(x) - \sum_{i=1}^{n+m} \lambda_i e_i(x) - \lambda_{n+m+1} \right) d\nu(x)$$

with variables ν and $\lambda \in \mathbf{R}^{n+m+1}$. We obtain the classic dual as a portfolio replication problem:

$$\begin{aligned} & \text{maximize} && \lambda_{n+m+1} + \sum_{i=1}^{n+m} \lambda_i p_i \\ & \text{subject to} && e_0(x) - \sum_{i=1}^{n+m} \lambda_i e_i(x) - \lambda_{n+m+1} \geq 0, \quad \text{for all } x \in \mathbf{R}_+^n. \end{aligned} \tag{17}$$

in the variable $\lambda \in \mathbf{R}^{n+m+1}$.

Unfortunately, the problem formulations above are numerically intractable except in certain particular cases (see [BP02] and [dEG03]). On the other hand, as we have seen in the previous section, the conditions of theorem 4 turn problem (1) into an infinite dimensional semidefinite program which can be relaxed to produce tractable bounds on the solution of (1). Here, we detail the accompanying duality theory to exhibit a static hedging portfolios corresponding to these bounds.

We can assume without loss of generality that the payoff functions $\{e_i(x)\}_{i=0, \dots, m+n}$, together with the cash $1_{\mathbb{S}}$, are linearly independent. Then [BCR84, Proposition 6.1.8 and Theorem 6.1.10] hold and we can form a dual to the cone of positive semidefinite functions on \mathbb{S} as follows. For $p \in \mathcal{A}(\mathbb{S})$ with $p = \sum_i q_i \chi_{s_i}$ and $f : \mathbb{S} \rightarrow \mathbf{R}$ with:

$$\langle f, p \rangle = \sum_i q_i f(s_i),$$

[BCR84, Theorem 6.1.10] states that Σ is the polar cone of \mathcal{P} for the above bilinear form, in other words:

$$\langle f, p \rangle \geq 0 \text{ for all } p \in \Sigma \text{ iff } f \in \mathcal{P}.$$

We can use this conic duality to compute a dual to program (13). Considering the compact case for simplicity, Corollary (7) states that the initial pricing problem:

$$\begin{aligned} & \text{minimize} && p_0 := \mathbf{E}_\nu[e_0(x)] \\ & \text{subject to} && \mathbf{E}_\nu[e_i(x)] = p_i, \quad i = 1, \dots, n+m, \end{aligned}$$

is equivalent to the following (infinite) semidefinite program:

$$\begin{aligned} & \text{minimize} && y_2 \\ & \text{subject to} && M(y) \succeq 0 \\ & && M(e_j y) \succeq 0, \quad \text{for } j = 0, \dots, n+m \\ & && M\left(\left(\beta - \sum_{k=0}^{n+m} e_k\right)y\right) \succeq 0 \\ & && y_{(j+2)} = p_j, \quad \text{for } j = 1, \dots, n+m \\ & && y_1 = 1. \end{aligned}$$

in the variable $y : \mathbb{S} \rightarrow \mathbf{R}$. We can form the Lagrangian:

$$\begin{aligned} L(y, \lambda, q) & := y_2 + (1 - y_1)\lambda_{n+m+1} + \sum_{j=1}^{n+m} (p_j - y_{(j+2)})\lambda_j - \langle y, q_0 \rangle \\ & \quad - \sum_{j=0}^{n+m} \langle e_j y, q_j \rangle - \langle (\beta - \sum_{k=0}^{n+m} e_k)y, q_{n+1} \rangle \end{aligned}$$

or again:

$$\begin{aligned} L(y, \lambda, q) & := y_2 + (1 - y_1)\lambda_{n+m+1} + \sum_{j=1}^{n+m} (p_j - y_{(j+2)})\lambda_j - \langle y, q_0 \rangle \\ & \quad - \sum_{j=0}^{n+m} \langle y, e_j q_j \rangle - \langle y, (\beta - \sum_{k=0}^{n+m} e_k)q_{n+1} \rangle \end{aligned}$$

in the variables $y : \mathbb{S} \rightarrow \mathbf{R}$, $\lambda \in \mathbf{R}^{n+m+1}$ and $q_j \in \Sigma$ for $j = 0, \dots, (n+1)$. We then get the dual as a portfolio problem:

$$\begin{aligned} & \text{maximize} && \sum_{j=1}^{n+m} p_j \lambda_j + \lambda_{n+m+1} \\ & \text{subject to} && e_0(x) - \sum_{j=1}^{n+m} \lambda_j e_j(x) - \lambda_{n+m+1} \\ & && = q_0(x) + \sum_{j=1}^{n+m} q_j(x) e_j(x) + (\beta - \sum_{k=0}^{n+m} e_k(x))q_{n+1}(x) \end{aligned} \tag{18}$$

in the variables $\lambda \in \mathbf{R}^{n+m+1}$ and $q_j \in \Sigma$ for $j = 0, \dots, (n+1)$.

The key difference between this portfolio problem and the one in (17) is that the (intractable) positivity constraint $e_0(x) - \sum_{i=1}^{n+m} \lambda_i e_i(x) - \lambda_{n+m+1} \geq 0$ in (17) is replaced by the tractable condition that this portfolio be written as a combination of sums of squares of polynomials in $\mathcal{A}(\mathbb{S})$. Such combinations can be constructed directly from the dual solution to the semidefinite program in (13), hence a numerical solution to the program in (13) provides both a price bound and an accompanying portfolio.

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