

Static versus dynamic arbitrage bounds on multivariate option prices

Alexandre d'Aspremont *

April 29, 2004

Abstract

We compare static arbitrage price bounds on basket calls, *i.e.* bounds that only involve buy-and-hold trading strategies, with the price range obtained within a multivariate generalization of the [BS73] model. While there is no gap between these two sets of prices in the univariate case, we observe here that contrary to our intuition about model risk for at-the-money calls, there is a somewhat large gap between model prices and static arbitrage prices, hence a similarly large set of prices on which a multivariate [BS73] model cannot be calibrated but where no conclusion can be drawn on the presence or not of a static arbitrage opportunity.

1 Introduction

In the classic unidimensional [BS73] framework, there is no gap between the range of model prices on one hand and the range of prices that create a buy-and-hold arbitrage opportunity on the other. This means in practice that if we can't extract a market implied volatility from the price of a call, then we know that there either an error in the data or a buy-and-hold arbitrage opportunity in the market that makes this price unviable on the long run.

The [BS73] call price formula is a strictly increasing function of the volatility and can be easily inverted, we also know from [LL00] for example that testing for the presence of a buy-and-hold arbitrage is equivalent to testing for positivity, monotonicity and convexity on call prices, which can be done by inspection. The unidimensional setting is then extremely favorable, since model calibration can be achieved at very little numerical cost, and a final answer is obtained as easily on the presence or not of buy-and-hold (or static) arbitrage opportunities. Because of these numerical properties and for consistency, call prices are always quoted in terms of their implied volatility.

Unfortunately, those key numerical properties are lost in a multivariate setting. As we will see below, calibrating a multivariate [BS73] model is a non trivial exercise and testing for the absence of static arbitrage between basket options becomes an NP-Hard problem (see [BP02]).

*EECS Dept., U.C. Berkeley, CA 94720. alexandre.daspremont@m4x.org

Equity derivatives markets don't make very frequent use of basket options beyond some elementary spread options. However, interest rate derivatives market volatility information is mostly concentrated in caps and swaptions, which can be seen as basket options on forward rates (see [Reb98] or [d'A03b] among others). While empirical evidence suggests that multivariate lognormal approximations to market models (see [BDB99] or [d'A03b]) calibrate very well to prices of caps and swaptions, a satisfactory joint model of correlation and smile features has yet to be designed. This means in practice that there remains a range of option prices on which these models cannot be calibrated but where no conclusion can be drawn on the presence or not of a static arbitrage. In this work, we try to quantify the magnitude of this range of prices.

The paper is organized as follows, in section two we show how to derive bounds on the price of a basket call within a multivariate lognormal model. In section three we detail various relaxation techniques to compute static arbitrage bounds on the price of baskets while section four details some numerical results.

2 Model price bounds

In this section, we compute the range of prices covered by a multivariate lognormal model when calibrated to a set of (liquid) market instruments. This is a hard numerical problem in general but excellent estimates of these bounds can be computed in a multivariate [BS73] model. We briefly describe the results of [d'A03b] in a simple equity setting, for more details on market model approximations and interest rate options pricing, we refer the reader to [Reb98], [BDB99] or [d'A03b].

In this setting, the dynamics of the assets F_t^i are given by:

$$dF_s^i = F_s^i \sigma^i dW_s,$$

where W_s is a n -dimensional Brownian motion and $\sigma^i \in \mathbf{R}^n$ for $i = 1, \dots, n$ are the volatility parameters. We shall denote by $X \in \mathbf{S}^n$ the corresponding covariance matrix, with $X_{ij} = \sigma^{iT} \sigma^j$. The sum of lognormally distributed assets is not lognormal, but we use a lognormal approximation to price basket calls. From [d'A03b], we know that the price of a basket call with payoff:

$$\left(\sum_{i=1}^n w_i F_T^i - K \right)^+$$

at time T , can be approximated by a [BS73] call price using an appropriate variance V_T such that:

$$C = BS(w^T F_t, K, T, V_T) = (w^T F_t) \mathcal{N}(h(V_T)) - K \mathcal{N}(h(V_T) - \sqrt{V_T}), \quad (1)$$

where $\mathcal{N}(x)$ is the CDF of the normal distribution:

$$h(V_T) = \frac{\left(\ln \left(\frac{w^T F_t}{K} \right) + \frac{1}{2} V_T \right)}{\sqrt{V_T}},$$

with

$$V_T = \mathbf{Tr}(\Omega X) T,$$

where $\Omega \in \mathbf{S}^n$ is the matrix given by $\Omega = \hat{w}\hat{w}^T$ and

$$\hat{w}_i = \frac{w_i F_t^i}{w^T F_t}.$$

This gives swaptions prices accurate to within 1-4 basis points in the Libor market model. Furthermore, a good estimate of the hedging tracking error can be computed by robustness argument (see [d'A02]).

Since the [BS73] formula is strictly increasing with its variance term V_T , computing bounds on the model price of a basket call given market price data on other baskets (with the same maturity) is equivalent to solving the following semidefinite program:

$$\begin{aligned} \text{max./min.} \quad & \mathbf{Tr}(\Omega_0 X) \\ \text{subject to} \quad & \mathbf{Tr}(\Omega_i X) = V_{T,i}, \quad i = 1, \dots, m \\ & X \succeq 0, \end{aligned} \tag{2}$$

in the variable $X \in \mathbf{S}^n$ with $V_{T,i}$ such that:

$$BS(w_i^T F_t, K_i, T, V_{i,T}) = p_i, \quad i = 1, \dots, m,$$

where p_i are the market prices of basket call options with weights w_i , maturity T and strike K_i . Note that we implicitly assume that all the options have the same maturity, and that, without loss of generality, the risk-free interest rate is zero (we compare prices in the forward market). Let us remark however that the multiperiod generalization has exactly the same format (see [d'A03b]).

This last semidefinite program can be solved very efficiently using algorithms such as the one by [Stu99], we refer the reader to [NN94] or [BV04] for further details. This means that given a certain number of prices of liquid market instruments, we can efficiently compute upper and lower bounds on the model price of another instrument.

If the market price of an instrument falls outside of these bounds, we then know that calibrating the model to this instrument together with the original set of prices will be infeasible. This means that there is a dynamic arbitrage opportunity if (and only if) the market dynamics follow that of the model. In practice however, this more often means either that the model dynamics are not rich enough to capture the market price features or that there is a problem with the market data set (liquidity, outliers, missing data, etc). In the next section, we discuss ways of refining this diagnostic.

3 Static arbitrage bounds

In the previous section, we computed bounds on basket option prices assuming that the underlying assets followed lognormal dynamics. In this section, we are looking for price bounds on options (given prices of other liquid options) without any assumption on the asset distribution. The range of prices covered will of course be much larger than in the lognormal case but option prices falling outside of this range generate buy-and-hold arbitrage opportunities, which are much more robust to liquidity issues than the model arbitrage detailed in the last section.

Let $p \in \mathbf{R}_+^m$, $K \in \mathbf{R}_+^m$, $w \in \mathbf{R}^n$, $w_i \in \mathbf{R}^n$, $i = 1, \dots, m$ and $K_0 \geq 0$. We consider here the problem of computing upper and lower bounds on the price of an European basket call option with strike K_0 and weight vector w_0 :

$$\begin{aligned} \min./\max. \quad & \mathbf{E}_\pi(w_0^T x - K_0)^+ \\ \text{subject to} \quad & \mathbf{E}_\pi(w_i^T x - K_i)^+ = p_i, \quad i = 1, \dots, m, \end{aligned} \quad (3)$$

with respect to all probability distributions π with support in \mathbf{R}_+^n on the asset price vector x , consistent with a given set of observed prices p_i of options on other baskets. Note that we again implicitly assume that all the options have the same maturity, and that, without loss of generality, the risk-free interest rate is zero (we compare prices in the forward market).

We know from [BP02] that the problem described in (3) is NP-Hard and in the next section we describe several relaxation techniques providing upper (resp. lower) bounds on the upper (resp. lower) bounds described by (3).

3.1 Linear programming relaxation

In dimension one we know that if $C(K)$ is a function giving the price of an option of strike K , then $C(K)$ must be positive, decreasing and convex. With $C(0) = S$, we have a set of *necessary* conditions on call prices for the absence of arbitrage.

In fact it is well known (see [LL00] or [BP02] among others) that these conditions are also *sufficient*, so there is no arbitrage between some given market call prices p_i if and only if there is a function $C(K)$ such that:

- $C(K)$ positive
- $C(K)$ decreasing
- $C(K)$ convex
- $C(K_i) = p_i$ and $C(0) = S$, $i = 1, \dots, m$.

These conditions are easily generalized to the multidimensional case as follows: given a set of market prices for basket calls $C(w_i, K_i) = p_i$ and suppose there is no arbitrage, then the function $C(w, K)$ must satisfy:

- $C(w, K)$ positive
- $C(w, K)$ decreasing in K , increasing in w
- $C(w, K)$ jointly convex in (w, K)
- $C(w_i, K_i) = p_i$ and $C(w, 0) = S$, $i = 1, \dots, m$.

The key difference is here that these conditions are *not sufficient*. Nevertheless, we can form a relaxation of problem (3) as:

$$\begin{aligned} \min./\max. \quad & C(w_0, K_0) \\ \text{subject to} \quad & C(w, K) \text{ positive} \\ & C(w, K) \text{ decreasing in } K, \text{ increasing in } w \\ & C(w, K) \text{ jointly convex in } (w, K) \\ & C(w_i, K_i) = p_i \text{ and } C(w, 0) = w^T S, \quad i = 1, \dots, m. \end{aligned} \quad (4)$$

In this form, this is an infinite dimensional linear program and not directly tractable. However, we can discretize (4) into a finite linear program by sampling the constraints on the data points (w_i, K_i) . We get the following program

$$\begin{aligned}
& \text{maximize/minimize } p_0 \\
& \text{subject to } \begin{aligned}
& g_i^T((w_j, K_j) - (w_i, K_i)) \leq p_j - p_i, \quad i, j = 0, \dots, m \\
& g_{i,j} \geq 0, -1 \leq g_{i,n+1} \leq 0, \quad i = 0, \dots, m, \quad j = 1, \dots, n \\
& g_i^T((w_i, K_i)) = p_i, \quad i = 0, \dots, m,
\end{aligned} \tag{5}
\end{aligned}$$

in the variables $p_0 \in \mathbf{R}_+$ and $g_i \in \mathbf{R}^{n+1}$ for $i = 0, \dots, m$. This is a linear program with $m \times (n + 1)$ variables and can be solved very efficiently. Furthermore, [dEG02] show that program (5) is in fact an *exact discretization* of program (4) and that an optimal solution to (4) can be constructed from that of (5).

This first relaxation technique allows us to get upper and lower bounds on the solution to (3) at a minimal numerical cost, [dEG02] show that these bounds are exact in some particular cases, but in general nothing can be guaranteed about their performance.

Necessary and sufficient conditions describing when a function can be written as the price

$$p = \mathbf{E}_\pi(w^T x - K)^+$$

of a basket call with weight w and strike K were derived by [HS90] in their investigation of production functions. They get the following result: a function can be written

$$C(w, K) = \int_{\mathbf{R}_+^n} (w^T x - K)^+ d\pi(x),$$

with $w \in \mathbf{R}_+^n$ and $K > 0$, if and only if:

- $C(w, K)$ is *convex* and *homogenous* of degree one
- for every $w \in \mathbf{R}_{++}^n$, we have $\lim_{K \rightarrow \infty} C(w, K) = 0$ and $\lim_{K \rightarrow 0^+} \frac{\partial C(w, K)}{\partial K} = -1$
- $F(w) = \int_0^\infty e^{-K} d\left(\frac{\partial C(w, K)}{\partial K}\right)$ belongs to $C_0^\infty(\mathbf{R}_+^n)$
- For some $\tilde{w} \in \mathbf{R}_+^n$, $(-1)^{k+1} D_{\xi_1} \dots D_{\xi_k} F(\lambda \tilde{w}) \geq 0$, for all positive integers k and $\lambda \in \mathbf{R}_{++}$ and all ξ_1, \dots, ξ_k in \mathbf{R}_+^n .

The key difficulty here is that the last two smoothness and monotonicity conditions are numerically intractable, so this result is of very little practical help in refining the conditions used in (5).

3.2 A moment approach

In this section, we look for ways of improving the relaxation technique in (5). NP hardness means that we can't hope to get a tractable set of necessary conditions to solve the problem exactly, here instead we look for additional conditions on prices that produce a sequence of successively tighter bounds on the solution to (3).

The integral transform approach above suggests a link to moment theory. In fact, as detailed in [d'A03a], numerically tractable conditions for the existence of a measure π such that $p = \mathbf{E}_\pi(w^T x - K)^+$ can be obtained by a generalization of Bernstein-Bochner type results to the payoff semigroup (see [BCR84] for a complete exposition). We briefly recall this construction below.

We suppose that the market is composed of cash and n underlying assets x_i for $i = 1, \dots, n$ with $x \in \mathbf{R}_+^n$. We suppose that the forward prices of the assets are known and given by p_i , for $i = 1, \dots, n$, hence w_i is the Euclidean basis and $K_i = 0$ for $i = 1, \dots, n$. In addition to these basic products, there are $m + 1$ basket *straddles* on the assets x , with payoff given by $|w_{n+i}^T x - K_{n+i}|$, $i = 1, \dots, m$. Because a straddle is obtained as the sum of a call and a put, we get the market price of straddles from those of basket calls and forward contracts by call-put parity and the static arbitrage problem on straddles is strictly equivalent to problem (3) since we always assume that forward prices are quoted in the market.

For simplicity, we will note these payoff functions e_i , for $i = 0, \dots, m + n$, with $e_0(x) = |w_0^T x - K_0|$, $e_i(x) = x_i$ for $i = 1, \dots, n$ and $e_{(n+j)}(x) = |w_j^T x - K_j|$ for $j = 1, \dots, m$. In what follows, we will focus on the commutative semigroup (\mathbb{S}, \cdot) generated by the payoffs $e_i(x)$ for $i = 0, \dots, m + n$, the cash $1_{\mathbb{S}}$ and their products.

$$\mathbb{S} = \{1, x_1, \dots, |w_m^T x - K_m|, x_1^2, \dots, x_i |w_j^T x - K_j|, \dots\} \quad (6)$$

Let us recall that a function $f : \mathbb{S} \rightarrow \mathbf{R}$ is called positive semidefinite iff for all finite families $\{s_i\}$ of elements of \mathbb{S} , the matrix with coefficients $f(s_i s_j)$ is positive semidefinite. We then get the following result from [d'A03a], suppose the asset distribution has compact support K and \mathbb{S} is the payoff semigroup defined above. A function $f(s) : \mathbb{S} \rightarrow \mathbf{R}$ can be represented as

$$f(s) = \mathbf{E}_\nu[s(x)], \quad \text{for all } s \in \mathbb{S}, \quad (7)$$

for some measure ν on K , and satisfies the price constraints in (3) if and only if:

- (i) $f(s)$ is positive semidefinite,
- (ii) $f(e_i s)$ is positive semidefinite for $i = 0, \dots, n + m$,
- (iii) $(\beta f(s) - \sum_{i=0}^{n+m} f(e_i s))$ is positive semidefinite,
- (iv) $f(e_i) = p_i$ for $i = 1, \dots, n + m$.

Furthermore, for each function f satisfying conditions (i) to (iv), the measure ν . A similar result holds in the case where the support of ν is not compact (see [d'A03a]).

The above result shows that testing for the absence of static arbitrage between the securities in \mathbb{S} , *i.e.* the set of straddles and their products, is equivalent to testing the positivity of an infinite number of matrices. This gives a direct recipe for writing a relaxation to problem (3): by considering a finite subset of these matrix inequalities, (3) can be turned into a semidefinite program. We summarize this procedure below.

We begin by recalling the construction of moment matrices in [Las01]. Let $\mathcal{A}(\mathbb{S})$ denote the real algebra generated by the payoffs in \mathbb{S} . We adopt the following multiindex notation for monomials in $\mathcal{A}(\mathbb{S})$:

$$e^\alpha(x) := e_0^{\alpha_0}(x) e_1^{\alpha_1}(x) \cdots e_{m+n}^{\alpha_{m+n}}(x),$$

and we let

$$y_e = (1, e_0, \dots, e_{m+n}, e_0^2, e_0e_1, \dots, e_0^d, \dots, e_{m+n}^d) \quad (8)$$

be the vector of all monomials in $\mathcal{A}(\mathbb{S})$, up to degree d , listed in graded lexicographic order. We note $s(d)$ the size of the vector y_e . Let $y \in \mathbf{R}^{s(2d)}$ be the vector of moments (indexed as in y_e) of some probability measure ν with support in \mathbf{R}_+^n , we note $M_d(y) \in \mathbf{R}^{s(d) \times s(d)}$, the symmetric matrix:

$$M_d(y)_{i,j} = \int_{\mathbf{R}_+^n} (y_e)_i(x) (y_e)_j(x) d\nu(x), \quad \text{for } i, j = 1, \dots, s(d).$$

In the rest of the paper, we will always implicitly assume that $y_1 = 1$. With $\beta(i)$ the exponent of the monomial $(y_\chi)_i$ and conversely, $i(\beta)$ the index of the monomial e^β in y_e . We notice that for a given moment vector $y \in \mathbf{R}^{s(d)}$ ordered as in (6), the first row and columns of the matrix $M_d(y)$ are then equal to y . The rest of the matrix is constructed according to:

$$M_d(y)_{i,j} = y_{i(\alpha+\beta)} \text{ if } M_d(y)_{i,1} = y_{i(\alpha)} \text{ and } M_d(y)_{1,j} = y_{i(\beta)}.$$

Similarly, let $g \in \mathcal{A}(\mathbb{S})$, we derive the moment matrix for the measure $g(x)d\nu$ on \mathbf{R}_+^n , noted $M_d(gy) \in \mathbf{S}^{s(d)}$, from the matrix of moments $M_d(y)$ by:

$$M_d(gy)_{i,j} = \int_{\mathbf{R}_+^n} (y_e)_i(x) (y_e)_j(x) g(x) d\nu(x) \quad \text{for } i, j = 1, \dots, s(d).$$

The coefficients of the matrix $M_m(gy)$ are then given by:

$$M_d(gy)_{i,j} = \sum_{\alpha} g_{\alpha} y_{i(\beta(i)+\beta(j)+\alpha)} \quad (9)$$

We can now form a semidefinite program to compute a lower bound on the optimal solution to (3) using a subset of the moment constraints in (7), taking only monomials and moments in y up to a certain degree. Let N be a positive integer and $y \in \mathbf{R}^{s(2N)}$, a lower bound on the optimal value of:

$$\begin{aligned} & \text{minimize} && p_0 := \mathbf{E}_{\nu}[e_0(x)] \\ & \text{subject to} && \mathbf{E}_{\nu}[e_i(x)] = p_i, \quad i = 1, \dots, n+m, \end{aligned}$$

can be computed as the solution of the following semidefinite program (again see [NN94] or [BV04]):

$$\begin{aligned} & \text{minimize} && y_2 \\ & \text{subject to} && M_N(y) \succeq 0 \\ & && M_N(e_j y) \succeq 0, \quad \text{for } j = 1, \dots, n, \\ & && M_N((\beta - \sum_{k=0}^{n+m} e_k) y) \succeq 0 \\ & && y_{(j+2)} = p_j, \quad \text{for } j = 1, \dots, n+m \text{ and } s \in \mathbb{S} \end{aligned} \quad (10)$$

where s is such that $i(s) \leq s(2N)$. As N increases, program (10) provides an increasingly precise relaxation of problem (3).

4 Numerical results

Here, we try to quantify on simple examples the magnitude of the gap between model prices, the price range obtained using the various relaxation techniques detailed above and the exact price bounds in (3).

4.1 Discrete model

Here, we simulate a set of arbitrage free basket call prices using a simple discrete model. Given these prices and the absence of arbitrage between basket calls, we study the price bounds induced on another basket call.

Of course we can't compare these bounds with the exact solution, however we can try to compare these bounds with *inner* bounds obtained by maximizing and minimizing the price $C(\omega_0, K_0)$ over a set of probability measures satisfying the price constraints $C(\omega_i, K_i) = p_i$. If we consider only discrete measures, this becomes a (very large) linear program.

We suppose here that the asset price at maturity T lies within the unit box $[0, 1]^n$. We then discretize the probability density using a grid with N bins per asset. The problem of finding (inner) upper and lower bounds on a basket (ω_0, K_0) can be written as:

$$\begin{aligned} \max./\min. \quad & \mathbf{E}_\nu (\omega_0^T x - K_0)_+ \\ \text{subject to} \quad & \mathbf{E}_\nu (\omega_i^T x - K_i)_+ = p_i, \quad i = 1, \dots, m, \end{aligned} \quad (11)$$

which is a linear program of (exponential) size N^n in the discrete measure ν .

We test these upper and lower bounds in dimension two. Program (11) can be written:

$$\begin{aligned} \max./\min. \quad & \sum_{k,l=0,\dots,N} \nu_{kl} (\omega_0^T [k/N, l/N]^T - K_0)_+ \\ \text{subject to} \quad & \sum_{k,l=0,\dots,N} \nu_{kl} (\omega_i^T [k/N, l/N]^T - K_i)_+ = p_i, \quad i = 1, \dots, m. \end{aligned}$$

which is a linear program in the variable $\nu \in \mathbf{R}^{N \times N}$. The assets are noted x_1, x_2 and we look for bounds on the price of an index option with payoff $(x_1 + x_2 - K)_+$. To produce price data, we use a simple discrete model for the assets, their distribution has finite support and is given by:

$$x = \{(0, 0), (0, .8), (.8, .3), (.6, .6), (.1, .4), (1, 1)\}$$

with probability

$$\{.2, .2, .2, .1, .1, .2\}.$$

The input data set is composed of the forward prices together with the following call prices:

$$\begin{aligned} & (.2x_1 + x_2 - .1)_+, (.5x_1 + .8x_2 - .8)_+, (.5x_1 + .3x_2 - .4)_+, \\ & (x_1 + .3x_2 - .5)_+, (x_1 + .5x_2 - .5)_+, (x_1 + .4x_2 - 1)_+, (x_1 + .6x_2 - 1.2)_+. \end{aligned}$$

We plot the inner and outer bounds obtained using this data in figure (1). We observe that sometimes the bounds match, *i.e.* the price bounds given by the relaxation are tight, while sometime there is a gap and not much can be said about the relaxation's suboptimality.

We also examine how these bounds evolve as more and more instruments are incorporated into the data set. For a particular choice of strike price (here $K = 1$), we compute the outer bounds (5) and inner bounds (11) obtained when using only the k first instruments in the data set, for $k = 2, \dots, 7$. The result is plotted in figure (2). We notice that it takes relatively few basket prices to get good bounds on the price of the target option.

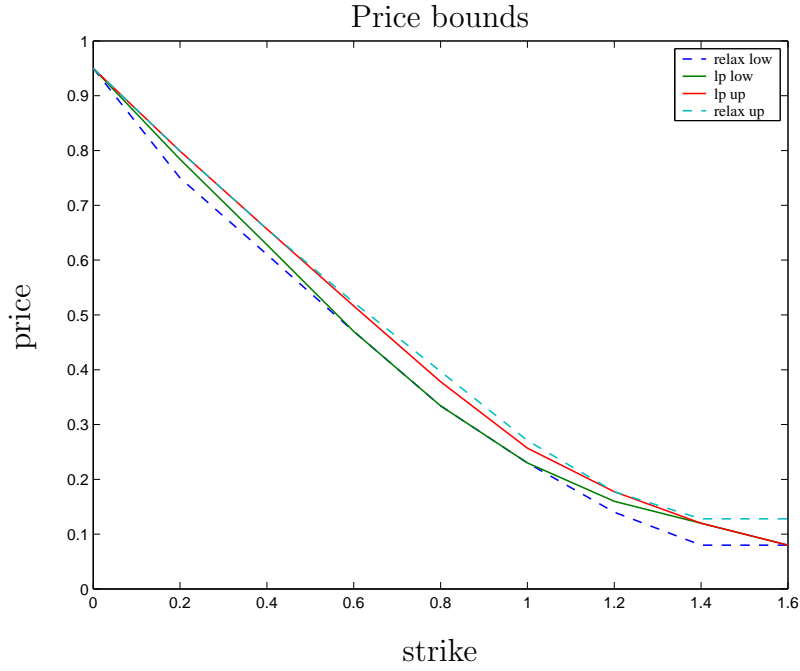


Figure 1: Comparison of the inner bounds computed by discretization (solid lines, computed using (11)) and the outer bounds obtained by relaxation (dashed lines, computed using (5)).

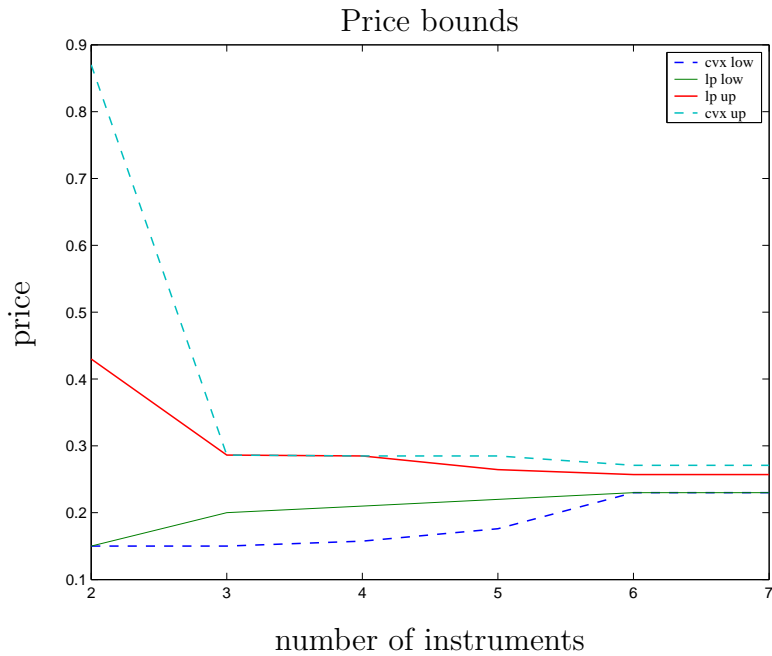


Figure 2: Inner bounds (solid lines, computed using (11)) and outer bounds (dotted lines, computed using (5)) versus number of instruments in the data set.

4.2 Multivariate lognormal model

Here again, we simulate a set of arbitrage free basket call prices using this time the multivariate [BS73] model detailed in section one. Given these prices and the absence of arbitrage between basket calls, we study the price bounds induced on another basket call.

We look for bounds on the price of an index option: $(.2 \sum_{i=1}^5 x_i - K)_+$, given the price of all at-the-money single asset calls and at-the-money basket calls with the following weights:

$$\begin{bmatrix} 0.33 & 0.33 & 0.33 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.33 & 0.33 & 0.33 \\ 0.40 & 0.20 & 0.20 & 0.20 & 0.00 \end{bmatrix}$$

The assets initial values F_t^i are $(0.03, 0.03, .05, .07, .07)$ and the model covariance matrix X is given by:

$$\begin{pmatrix} 0.06 & 0.04 & 0.04 & 0.04 & 0.04 \\ 0.04 & 0.06 & 0.04 & 0.04 & 0.04 \\ 0.04 & 0.04 & 0.06 & 0.04 & 0.04 \\ 0.04 & 0.04 & 0.04 & 0.06 & 0.04 \\ 0.04 & 0.04 & 0.04 & 0.04 & 0.06 \end{pmatrix}$$

We plot the inner and outer bounds obtained using this data in figure (3). Since the [BS73] of an at-the-money call is not very sensitive to the volatility, we could have expected the call price near the money to be somewhat insensitive to model specification. The fact that the gap between the inner model price bounds and the outer static arbitrage bounds is so large seems to directly contradict this intuition, showing that in fact the range of arbitrage free prices for basket calls is much larger than the range of prices that can be attained by a multivariate [BS73] model.

4.3 Moment constraints

For numerical reasons, we consider a model with two assets x_1, x_2 and look for bounds on the price of the basket $|x_1 + x_2 - K|$. We use a simple discrete model for the assets:

$$x = \{(0, 0), (0, 3), (3, 0), (1, 2), (5, 4)\}$$

, with probability

$$p = (.2, .2, .2, .3, .1)$$

, to simulate market prices for the forwards and the following straddles:

$$|x_1 - .9|, |x_1 - 1|, |x_2 - 1.9|, |x_2 - 2|, |x_2 - 2.1|.$$

The results are detailed in figure (4). Unfortunately the code from [Stu99], although very robust in general, is not as stable on the moment problems detailed here. A different implementation using large scale optimization techniques such as spectral bundle should prove more appropriate here.

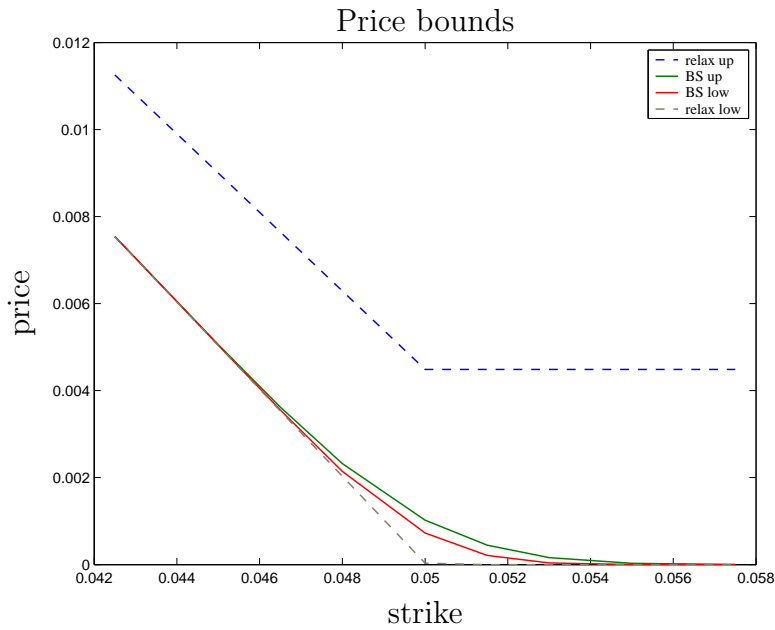


Figure 3: Comparison of the inner bounds computed using the Black & Scholes model (solid lines, computed using (2)) and the outer bounds obtained by relaxation (dashed lines, computed using (5)).

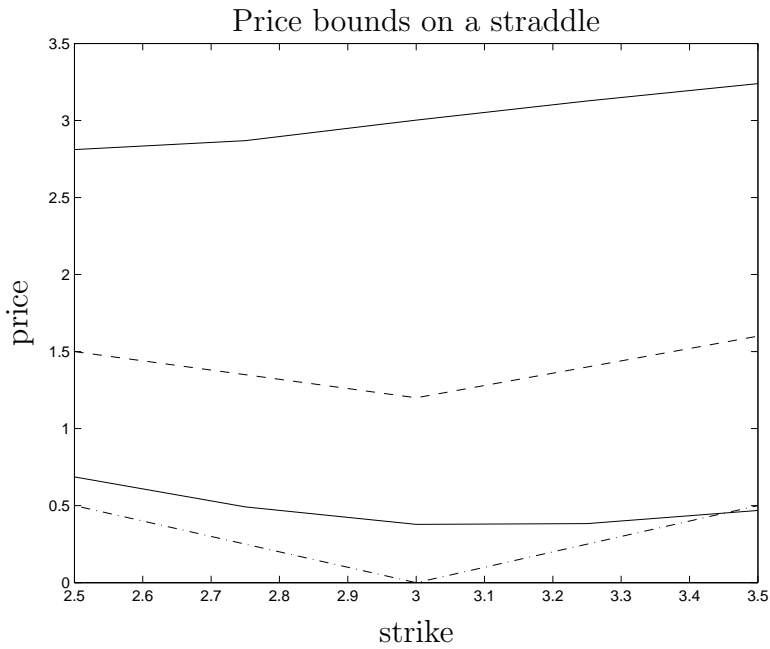


Figure 4: Upper and lower price bounds on a straddle (solid lines). The dashed lines represent the payoff function and the actual price.

5 Conclusion

We have detailed some tractable relaxation procedures to test for the absence of static arbitrage between basket calls. A comparison with the range of prices attained by a multivariate [BS73] model shows that this simple model only covers a relatively small portion of the range of arbitrage free prices. Running the same kind of test on a model with a richer smile structure, if numerically feasible, would be very interesting. Also, the numerical issues encountered in the moment relaxation technique are very surprising and probably deserve more investigation.

References

- [BCR84] Christian Berg, Jens Peter Reus Christensen, and Paul Ressel, *Harmonic analysis on semigroups : theory of positive definite and related functions*, Graduate texts in mathematics, vol. 100, Springer-Verlag, New York, 1984.
- [BDB99] Alan Brace, Tim Dun, and Geoff Barton, *Towards a central interest rate model*, Working Paper. FMMA (1999).
- [BP02] D. Bertsimas and I. Popescu, *On the relation between option and stock prices: a convex optimization approach*, *Operations Research* **50** (2002), no. 2, 358–374.
- [BS73] F. Black and M. Scholes, *The pricing of options and corporate liabilities*, *Journal of Political Economy* **81** (1973), 637–659.
- [BV04] S. Boyd and L. Vandenberghe, *Convex optimization*, Cambridge University Press, 2004.
- [d'A02] Alexandre d'Aspremont, *Interest rate model calibration and risk-management using semidefinite programming*, Ph.D. thesis, Ecole Polytechnique, 2002.
- [d'A03a] ———, *A harmonic analysis solution to the static basket arbitrage problem*, ArXiv: math.OC/0309048 (2003).
- [d'A03b] ———, *Interest rate model calibration using semidefinite programming*, *Applied Mathematical Finance* **10** (2003), no. 3, 183–213.
- [dEG02] Alexandre d'Aspremont and Laurent El Ghaoui, *Static arbitrage bounds on basket option prices*, EECS Technical report, UC Berkeley. ArXiv: math.OC/0302243 (2002).
- [HS90] G. Henkin and A. Shananin, *Bernstein theorems and Radon transform, application to the theory of production functions*, American Mathematical Society: Translation of mathematical monographs **81** (1990), 189–223.
- [Las01] J. B. Lasserre, *Global optimization with polynomials and the problem of moments*, *SIAM Journal on Optimization* **11** (2001), no. 3, 796–817.

- [LL00] J.P. Laurent and D. Leisen, *Building a consistent pricing model from observed option prices*, Quantitative Analysis in Financial Markets (M. Avellaneda, ed.), World Scientific Publishing, 2000.
- [NN94] I. Nesterov and A. Nemirovskii, *Interior-point polynomial algorithms in convex programming*, Society for Industrial and Applied Mathematics, Philadelphia, 1994.
- [Reb98] Riccardo Rebonato, *Interest-rate options models*, Financial Engineering, Wiley, 1998.
- [Stu99] Jos F. Sturm, *Using sedumi 1.0x, a matlab toolbox for optimization over symmetric cones*, Optimization Methods and Software **11** (1999), 625–653.