A Market Test for the Positivity of Arrow-Debreu Prices

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Introduction

- Classic Black & Scholes (1973) option pricing based on:
  - a *dynamic hedging* argument
  - a *model* for the asset dynamics (geometric BM)
- Sensitive to liquidity, transaction costs, model risk ...
- What can we say about derivative prices with much weaker assumptions?

Static Arbitrage

Here, we rely on a *minimal set of assumptions*:

- no assumption on the asset distribution
- one period model

An arbitrage in this simple setting is a *buy and hold* strategy:

- form a portfolio at no cost today with a strictly positive payoff at maturity
- no trading involved between today and the option's maturity

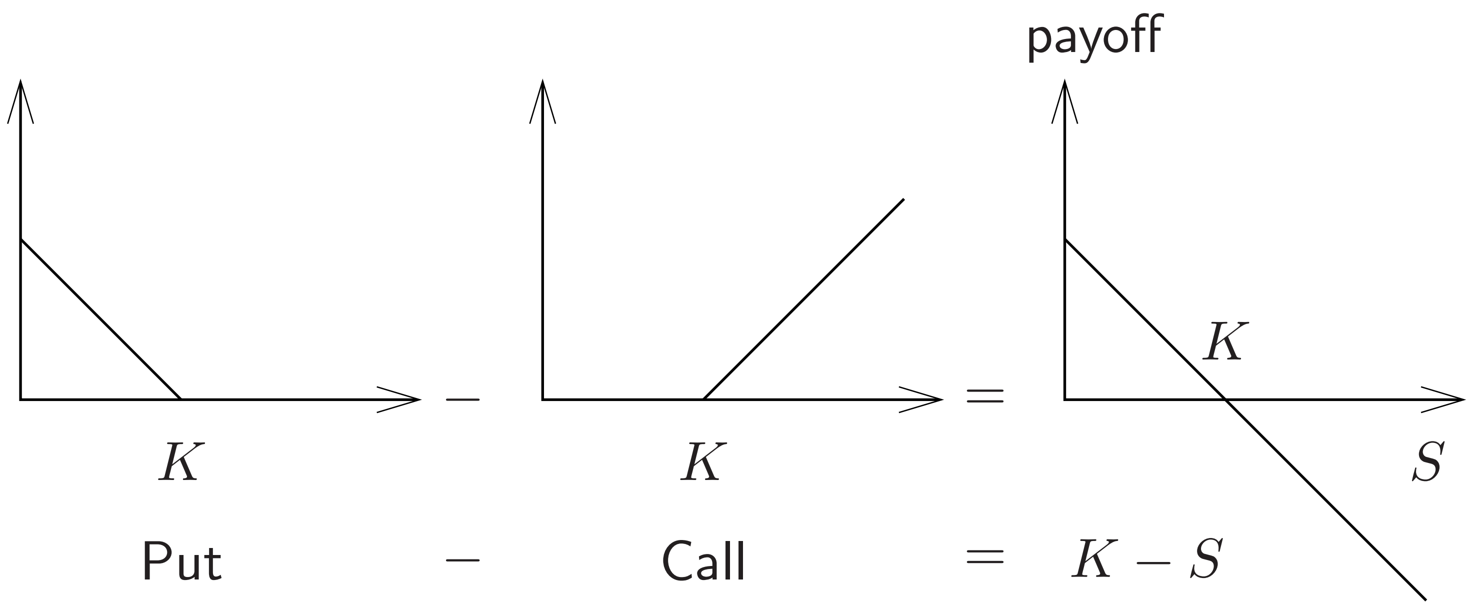
What for?

- Data validation (e.g. before calibration), static arbitrage means market data is incompatible with *any* dynamic model. . .
- Test extrapolation formulas
- In illiquid markets, find optimal static hedge

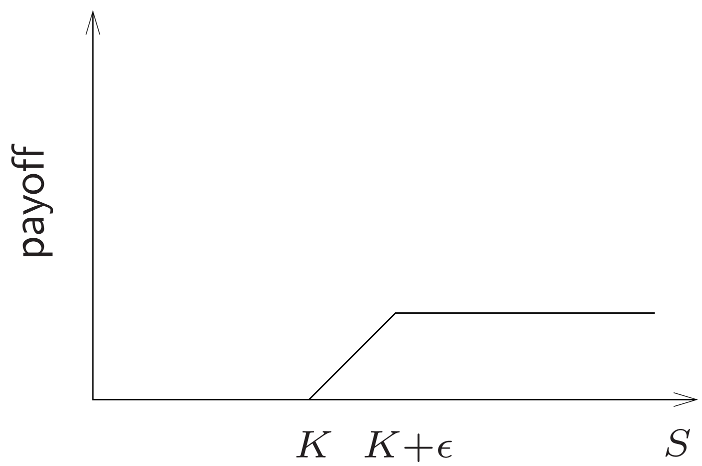
Outline

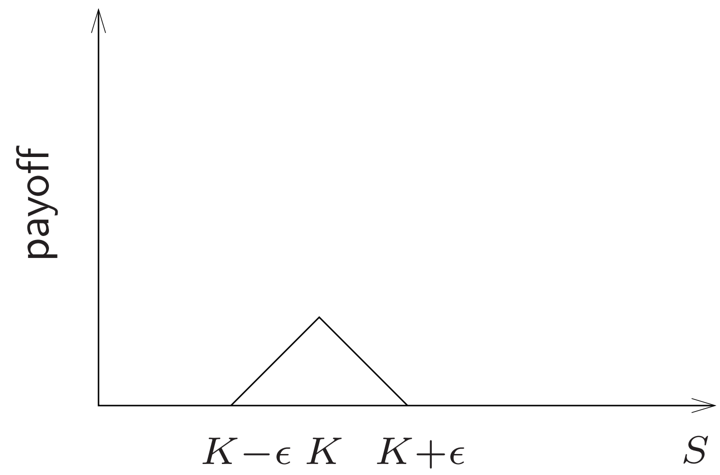
- **Static Arbitrage**
- Harmonic Analysis on Semigroups
- No Arbitrage Conditions

Simplest Example: Put Call Parity



Call Spread - Butterfly Spread





Here, the absence of arbitrage implies that the price of a call spread be positive, hence call prices must be *decreasing* with strike

$$C(K+\epsilon) - C(K) \leq 0$$

it also implies that the price of a butterfly spread be positive, and call prices must then be *convex* with strike

$$C(K+\epsilon) - 2C(K) + C(K-\epsilon) \geq 0$$

Price Constraints

The absence of arbitrage implies that if $C(K)$ is a function giving the price of an option of strike K, then $C(K)$ must satisfy:

- $C(K)$ positive
- $C(K)$ decreasing
- $C(K)$ convex

With $C(0) = S$, we have a set of *necessary* conditions for the absence of arbitrage

Sufficient Conditions

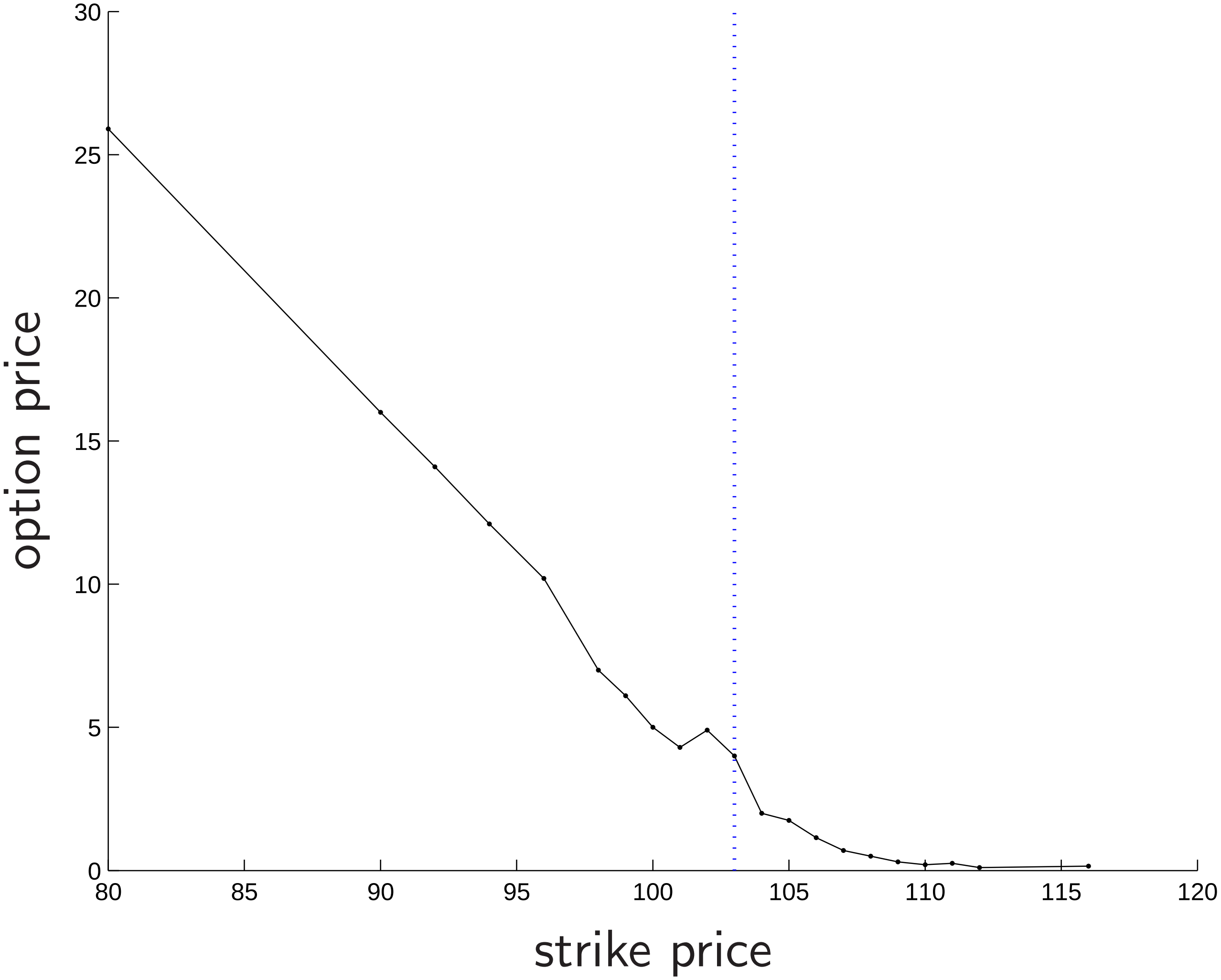
In fact, these conditions are also *sufficient*. . .

Suppose we have a set of market prices for calls $C(K_i) = p_i$, *then there is no arbitrage iff there is a function C(K):*

- $C(K)$ *positive*
- $C(K)$ *decreasing*
- $C(K)$ *convex*
- $C(K_i) = p_i$ *and* $C(0) = S$

This is *very easy* to test. . .

Dow Jones index call option prices on Mar. $17\ 2004$, maturity Apr. $16\ 2004$



Source: Reuters.

Why?

Data quality...

- All the prices are last quotes (not simultaneous)
- Low volume
- Some transaction costs

Problem: this data is used to calibrate models and price other derivatives...

Dimension n: Basket Options

- A basket call payoff is given by:

$$\left(\sum_{i=1}^{k} w_i S_i - K \right)^+$$

  where $w_1, \ldots, w_k$ are the basket's weights and K is the option's strike price

- Examples include: Index options, spread options, swaptions...

- Basket option prices are used to gather information on *correlation*

We denote by $C(w, K)$ the price of such an option, can we get conditions to test basket price data?

Necessary Conditions

Similar to dimension one...

Suppose we have a set of market prices for calls $C(w_i, K_i) = p_i$, and there is no arbitrage, then the function $C(w, K)$ satisfies:

- $C(w, K)$ positive
- $C(w, K)$ decreasing in K, increasing in w
- $C(w, K)$ jointly convex in (w, K)
- $C(w_i, K_i) = p_i$ and $C(0) = S$

This is still *tractable* in dimension n as a *linear program*.

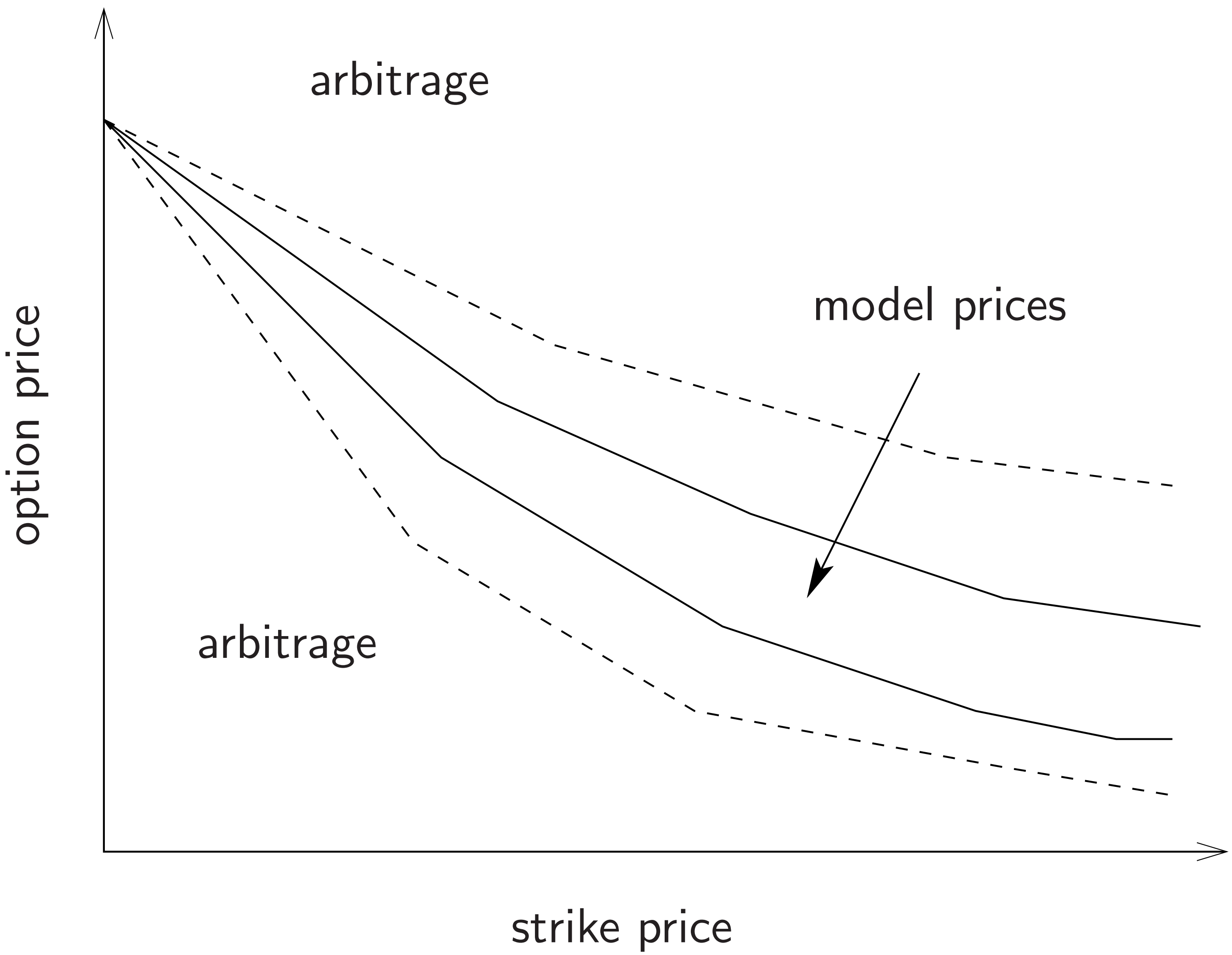
Sufficient?

A key difference with dimension one: Bertsimas & Popescu (2002) show that the exact problem is NP-Hard.

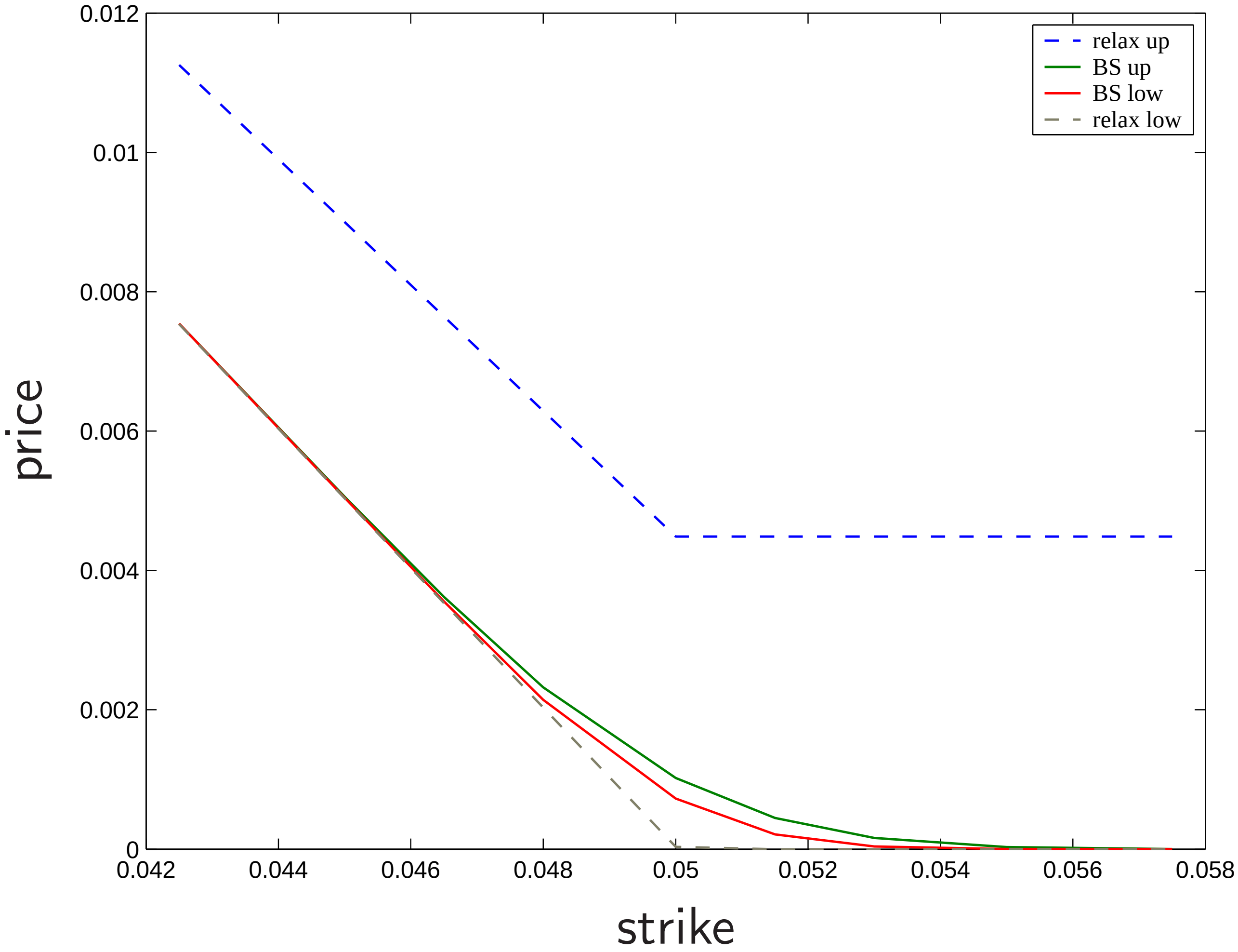
- These conditions are *only necessary*...
- Numerical cost is minimal (small LP)
- We can show *sufficiency* in some particular cases

Question: How tight are these conditions in general? Can we refine them?

Price Bounds



Multivariate Black-Scholes Model



Close the Gap

The gap is surprisingly large. . .

- ATM prices are not supposed to be very sensitive to the smile
- Approx. lognormal model calibrate easily to swaption data in practice

How can we improve the static bounds (so we know when to blame the model)?

Arrow-Debreu prices

- *Arrow-Debreu*: There is no arbitrage in the static market iff there is a probability measure π such that:

$$C(w, K) = \mathbf{E}_\pi (w^T x - K)^+$$

- $\pi(x)$ represents the price of the Arrow-Debreu security corresponding to state x

- Discretize on a uniform grid: This turns this into a *linear program* with m^n variables, where n is the number of assets x_i and m is the number of bins.

- Numerically: hopeless. . .

Integral Transform Solution

- For a given measure π, we can write basket call prices as:

$$\begin{aligned} C(w,K) &= \mathbf{E}_\pi (w^T x - K)^+ \\ &= \int_{\mathbf{R}^n_+} (w^T x - K)^+ d\pi(x), \end{aligned}$$

and think of $C_\pi(w,K)$ as a particular integral transform of π

- At least formally, we have:

$$\frac{\partial^2 C(w,K)}{\partial K^2} = \int_{\mathbf{R}^n_+} \delta(w^T x - K)\pi(x)dx$$

- This means that $\partial^2 C(w,K)/\partial K^2$ is the *Radon transform* (see Helgason (1999) or Ramm & Katsevich (1996)) of the measure π

A Range Characterization Problem...

- The general arbitrage problem can written as the following infinite dimensional problem:

$$\begin{array}{ll} \text{find} & C(w,K) \\ \text{subject to} & C(w_i,K_i) = p_i, \quad i = 1,\ldots,m \\ & C(w,K) \in \mathcal{R}_C, \end{array}$$

- Here, $\mathcal{R}_C$ is the range of the (linear) integral transform

$$\begin{aligned} C: \quad & \mathcal{K} \rightarrow \mathcal{R}_C \\ & \pi \rightarrow C(w,K) = \int_{\mathbf{R}_+^n} (w^T x - K)^+ d\pi(x) \end{aligned}$$

Question: How do *characterize* $\mathcal{R}_C$?

Full Conditions

derived by Henkin & Shananin (1990). A function can be written

$$C(w,K) = \int_{\mathbf{R}^n_+} (w^T x - K)^+ d\pi(x)$$

with $w \in \mathbf{R}^n_+$ and $K > 0$, if and only if:

- $C(w,K)$ is *convex* and *homogenous* of degree one;
- $\lim_{K\to\infty} C(w,K) = 0$ and $\lim_{K\to 0^+} \frac{\partial C(w,K)}{\partial K} = -1$
- $F(w) = \int_0^\infty e^{-K} d\left(\frac{\partial C(w,K)}{\partial K}\right)$ belongs to $C_0^\infty(\mathbf{R}^n_+)$
- For some $\tilde{w} \in \mathbf{R}^n_+$ the inequalities: $(-1)^{k+1} D_{\xi_1}...D_{\xi_k} F(\lambda\tilde{w}) \geq 0$, for all positive integers k and $\lambda \in \mathbf{R}_{++}$ and all $\xi_1,\ldots,\xi_k$ in $\mathbf{R}^n_+$.

Finer Conditions

- The convexity condition appears again here. . .

- The last two conditions (smoothness and total positivity) in the Radon range characterization are hard to implement, yet they suggest a *moment approach*. . .

- Can we find *tractable* conditions?

Tractable Conditions

- Bochner's theorem on the Fourier transform of positive measures:

$$f(s) = \int e^{-i<s,x>} g(x) dx \quad \text{with } g(x) \geq 0$$

$$\Updownarrow$$

$$f(s) \text{ positive semidefinite}$$

which means testing if the *matrices* $f(s_i s_j)$ are positive semidefinite

- Can we generalize this result to other transforms? In particular:

$$\int_{\mathbf{R}_+^n} (w^T x - K)^+ d\pi(x)$$

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- **Harmonic Analysis on Semigroups**
- No Arbitrage Conditions

Harmonic Analysis on Semigroups

Some quick definitions...

- A pair $(\mathbb{S}, \cdot)$ is called a *semigroup* iff:
  - if $s, t \in \mathbb{S}$ then $s \cdot t$ is also in $\mathbb{S}$
  - there is a neutral element $e \in \mathbb{S}$ such that $e \cdot s = s$ for all $s \in \mathbb{S}$

- The *dual* $\mathbb{S}^*$ of $\mathbb{S}$ is the set of *semicharacters*, $i.e.$ applications $\chi : \mathbb{S} \to \mathbf{R}$ such that
  - $\chi(s)\chi(t) = \chi(s \cdot t)$ for all $s, t \in \mathbb{S}$
  - $\chi(e) = 1$, where e is the neutral element in $\mathbb{S}$

- A function α is called an *absolute value* on $\mathbb{S}$ iff
  - $\alpha(e) = 1$
  - $\alpha(s \cdot t) \leq \alpha(s)\alpha(t)$, for all $s, t \in \mathbb{S}$

Harmonic Analysis on Semigroups

Last definitions (honest)...

- A function $f : \mathbb{S} \to \mathbf{R}$ is *positive semidefinite* iff for every family $\{s_i\} \subset \mathbb{S}$ the matrix with elements $f(s_i \cdot s_j)$ is positive semidefinite

- A function f is *bounded* with respect to the absolute value α iff there is a constant $C > 0$ such that

$$|f(s)| \leq C\alpha(s), \quad s \in \mathbb{S}$$

- f is *exponentially bounded* iff it is bounded with respect to an absolute value

Harmonic Analysis on Semigroups: Central Result

The central result, see Berg, Christensen & Ressel (1984) based on Choquet's theorem:

- the set of exponentially bounded *positive definite functions* is a *Bauer simplex* whose extreme points are the bounded semicharacters...

- this means that we have the following representation for positive definite functions on $\mathbb{S}$:
$$f(s) = \int_{\mathbb{S}^*} \chi(s) d\mu(\chi)$$
where μ is a Radon measure on $\mathbb{S}^*$

Harmonic Analysis on Semigroups: Simple Examples

- *Berstein's theorem* for the Laplace transform

$$\mathbb{S} = (\mathbf{R}_+, +),\ \chi_x(t) = e^{-xt} \text{ and } f(t) = \int_{\mathbf{R}_+} e^{-xt} d\mu(x)$$

- with involution, *Bochner's theorem* for the Fourier transform

$$\mathbb{S} = (\mathbf{R}, +),\ \chi_x(t) = e^{2\pi ixt} \text{ and } f(t) = \int_{\mathbf{R}} e^{2\pi ixt} d\mu(x)$$

- *Hamburger's solution* to the unidimensional moment problem

$$\mathbb{S} = (\mathbf{N}, +),\ \chi_x(k) = x^k \text{ and } f(k) = \int_{\mathbf{R}} x^k d\mu(x)$$

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The Option Pricing Problem Revisited

- the basket option payoffs $(w^T x - K)^+$ are not ideal in this setting
- solution, use *straddles*: $|w^T x - K|$
- as straddles are just the *sum of a call and a put*, their price can be computed from that of the corresponding call and forward by call-put parity
- the fact that $|w^T x - K|^2$ is a polynomial keeps the complexity low

Payoff Semigroup

- the fundamental semigroup $\mathbb{S}$ is here the multiplicative *payoff semigroup* generated by the cash, the forwards and the straddles:

$$\mathbb{S} = \{1, x_1, \ldots, x_n, |w_1^T x - K_1|, \ldots, |w_m^T x - K_m|, x_1^2, x_1 x_2, \ldots\}$$

- the *semicharacters* are the functions $\chi_x : \mathbb{S} \rightarrow \mathbf{R}$ which evaluate the payoffs at a certain point x

$$\chi_x(s) = s(x), \quad \text{for all } s \in \mathbb{S}$$

The Option Pricing Problem Revisited

- the original static arbitrage problem can be reformulated as

$$\begin{array}{ll} \text{find} & f \\ \text{subject to} & f(|w_i^T x - K_i|) = p_i, \quad i = 1, \ldots, m \\ & f(s) = \mathbf{E}_\pi[s], \quad s \in \mathbb{S} \quad \text{(f moment function)} \end{array}$$

- the variable is now $f : \mathbb{S} \rightarrow \mathbf{R}$, a function that associates to each payoff s in $\mathbb{S}$, its price $f(s)$

- the representation result in Berg et al. (1984) shows when a (price) function $f : \mathbb{S} \rightarrow \mathbf{R}$ can be represented as

$$f(s) = \mathbf{E}_\pi[s]$$

Option Pricing: Main Theorem

If we assume that the asset distribution has a compact support included in $\mathbf{R}_+^n$, and note e_i for $i = 1, \ldots, n+m$ the forward and option payoff functions we get:

A function $f(s) : \mathbb{S} \rightarrow \mathbf{R}$ *can be represented as*

$$f(s) = \mathbf{E}_\nu[s(x)], \quad \textit{for all } s \in \mathbb{S},$$

for some measure ν *with compact support, iff for some* $\beta > 0$*:*

(i) $f(s)$ *is positive semidefinite*

(ii) $f(e_i s)$ *is positive semidefinite for* $i = 1, \ldots, n+m$

(iii) $\left(\beta f(s) - \sum_{i=1}^{n+m} f(e_i s)\right)$ *is positive semidefinite*

this turns the basket arbitrage problem into a *semidefinite program*

Semidefinite Programming

A *semidefinite program* is written:

$$\begin{array}{ll} \text{minimize} & \mathbf{Tr}\, CX \\ \text{subject to} & \mathbf{Tr}\, A_i X = b_i, \quad i = 1, \ldots, m \\ & X \succeq 0, \end{array}$$

in the variable $X \in \mathbf{S}^n$, with parameters $C, A_i \in \mathbf{S}^n$ and $b_i \in \mathbf{R}$ for $i = 1, \ldots, m$. Its *dual* is given by:

$$\begin{array}{ll} \text{maximize} & b^T \lambda \\ \text{subject to} & C - \sum_{i=1}^m \lambda_i A_i \succeq 0, \end{array}$$

in the variable $\lambda \in \mathbf{R}^m$.

Extension of interior point techniques for linear programming show how to solve these convex programs *efficiently* (see Nesterov & Nemirovskii (1994), Sturm (1999) and Boyd & Vandenberghe (2004)).

Option Pricing: a Semidefinite Program

we get a relaxation by only sampling the elements of $\mathbb{S}$ up to a certain degree, the variable is then the vector $f(s)$ with

$$e = (1, x_1, \ldots, x_n, |w_1^T x - K_1|, \ldots, |w_m^T x - K_m|, x_1^2, x_1 x_2, \ldots, |w_m^T x - K_m|^N)$$

testing for the absence of arbitrage is then a *semidefinite program*:

$$\begin{array}{ll} \text{find} & f \\ \text{subject to} & M_N(f(s)) \succeq 0 \\ & M_N(f(e_j s)) \succeq 0, \quad \text{for } j = 1, \ldots, n, \\ & M_N\left(f((\beta - \sum_{k=1}^{n+m} e_k)s)\right) \succeq 0 \\ & f(e_j) = p_j, \quad \text{for } j = 1, \ldots, n+m \text{ and } s \in \mathbb{S} \end{array}$$

where $M_N(f(s))_{ij} = f(s_i s_j)$ and $M_N(f(e_k s))_{ij} = f(e_k s_i s_j)$

Duality

- the price maximization program is:

$$\begin{array}{ll} \text{maximize} & \int_{\mathbf{R}_+^n} (w_0^T x - K_0)^+ \pi(x) dx \\ \text{subject to} & \int_{\mathbf{R}_+^n} (w_i^T x - K_i)^+ \pi(x) dx = p_i, \quad i = 1, \ldots, m \\ & \int_{\mathbf{R}_+^n} \pi(x) dx = 1, \end{array}$$

  in the variable $\pi \in \mathcal{K}$.

- the dual is a *portfolio problem*:

$$\begin{array}{ll} \text{minimize} & \lambda^T p + \lambda_0 \\ \text{subject to} & \sum_{i=1}^m \lambda_i (w_i^T x - K_i)^+ + \lambda_0 \geq \psi(x) \text{ for every } x \in \mathbf{R}_+^n \end{array}$$

  in the variable $\lambda \in \mathbf{R}^{m+1}$.

very intuitive, but completely intractable. . .

Conic Duality

let $\Sigma \subset \mathcal{A}(\mathbb{S})$ be the set of polynomials that are sums of squares of polynomials in $\mathcal{A}(\mathbb{S})$, and $\mathcal{P}$ the set of positive semidefinite sequences on $\mathbb{S}$

- instead of the conic duality between probability measures and positive portfolios

$$p(x) \geq 0 \Leftrightarrow \int p(x) d\nu \geq 0, \quad \text{for all measures } \nu$$

- we use the duality between positive semidefinite sequences $\mathcal{P}$ and sums of squares polynomials Σ

$$p \in \Sigma \Leftrightarrow \langle f, p \rangle \geq 0 \text{ for all } f \in \mathcal{P}$$

with $p = \sum_i q_i \chi_{s_i}$ and $f : \mathbb{S} \to \mathbf{R}$, where $\langle f, p \rangle = \sum_i q_i f(s_i)$

Option Pricing: Dual

- the dual of the price maximization problem

$$
\begin{array}{ll}
\text{maximize} & f(e_0) \\
\text{subject to} & M_N(f(s)) \succeq 0 \\
& M_N(f(e_j s)) \succeq 0, \quad \text{for } j = 1, \ldots, n, \\
& M_N\left(f((\beta - \sum_{k=1}^{n+m} e_k)s)\right) \succeq 0 \\
& f(e_j) = p_j, \quad \text{for } j = 1, \ldots, n+m \text{ and } s \in \mathbb{S}
\end{array}
$$

- now becomes...

$$
\begin{array}{ll}
\text{minimize} & \sum_{j=1}^{n+m} p_j \lambda_j + \lambda_{n+m+1} \\
\text{subject to} & \sum_{j=1}^{n+m} \lambda_j e_j(x) + \lambda_{n+m+1} - |w_0^T x - K_0| \\
& = q_0(x) + \sum_{j=1}^{n+m} q_j(x) e_j(x) + (\beta - \sum_{k=0}^{n+m} e_k(x)) q_{n+1}(x)
\end{array}
$$

in the variables $\lambda \in \mathbf{R}^{n+m+1}$ and $q_j \in \Sigma$ for $j = 0, \ldots, (n+1)$

Option Pricing: Caveats

- *Size*: grows exponentially with the number of assets: no free lunch. . .

- In dimension 2, for spread options, this is:

$$\binom{2+d}{2}(k+1)$$

  where d is the degree of the relaxation and k the number of assets.

- Conditioning issues. . .

Conclusion

- Testing for static arbitrage in option price data is easy in dimension one
- The extension on basket options (swaptions, etc) is NP-hard but good relaxations can be found
- We get a computationally friendly set of conditions for the absence of arbitrage
- Small scale problems are tractable in practice as semidefinite programs

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