Synchronization and Related Phenomena in Networks of Diffusively-Coupled Fitzhugh-Nagumo Oscillators

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Motivation and Background
- Synchronized activity is crucial for brain function:
  - Occurs at multiple levels (basal ganglia, local field potential)
  - Related to many pathological conditions (epilepsy)
- Insight about synchronization can lead to advances in:
  - Deep Brain Stimulation, Transcranial Stimulation
  - System Identification
  - Testable predications
  - Measurable efficacy metrics for disease treatment

For a network of N-oscillators with state $x_i \in \mathbb{R}^n$, 1 ≤ $i$ ≤ N we define the synchronization manifold as $\mathcal{S} = \{x_1 = x_2 = \cdots = x_N | X^t \in \mathbb{R}^n\}$. Then the network synchronizes completely if the state trajectories converge to $\mathcal{S}$ in some appropriate norm.

Cluster synchronization in a network of non-identical oscillators
For a network of N-oscillators with state $x_i \in \mathbb{R}^n$, 1 ≤ $i$ ≤ N we define the cluster synchronization manifold as $\mathcal{S}^K = \{x_1 = \cdots = x_n = x_{N-k+1} = \cdots = x_N | x_i \in \mathbb{R}^n\}$, where 1 ≤ $k$ ≤ N, and there exists 1 ≤ $c_1$, $c_2$, $c_k$ ≤ N such that $c_1 + \cdots + c_k = N$. Then the network synchronizes in clusters if the state trajectories converge to $\mathcal{S}^K$ in some appropriate norm. The $k$-th cluster is defined as $\mathcal{C}^k = \{k-1 \sum_{i=1}^{k-1} c_i + 1, \sum_{i=1}^{k-1} c_i + 2, \cdots, \sum_{i=1}^{k-1} c_i\}$.

- Sufficient Condition:
  $\gamma > \frac{1 + \alpha_k}{\lambda_2(\mathcal{G}) + \lambda_2(\mathcal{G})}$, $\forall 1 \leq k \leq K$, $\alpha_k = \frac{(\epsilon(k) - 1/p)^2}{4b(\epsilon(k))}$, $p = \max_{1 \leq k \leq K} \frac{1}{\lambda^2_i}$

Necessity of cluster-input-equivalence condition
- Oscillators belonging to the same cluster (e.g. $C^k$) are identical.
- Cluster-input-Equivalence Condition ($k$-th cluster): $\sum_{m \in \mathcal{C}^k} \gamma_{im} = \sum_{m \in \mathcal{C}^k} \gamma_{jm}, \forall m \in \{1, \cdots, K\} \setminus k, \forall i, j \in \mathcal{C}^k$

Fitzhugh-Nagumo (FN) neuronal oscillator
- Fast Dynamics (Membrane Potential):
  $\dot{y} = -y^3/3 - z + \epsilon$
- Slow Dynamics (Recovery Variable):
  $\dot{z} = (y - bz + a)$

The dynamics of a FN Oscillator are strictly semi-passive, i.e. outside a ball around the origin, it behaves as a strictly passive system.

Diffusively-coupled network of FN oscillators

- Individual dynamics:
  $\dot{y}_i = y_i - y_i^3/3 - z_i + I_i + u_i$ $\dot{z}_i = \epsilon(y_i - bz_i + a_i)$ $\dot{u}_i = \sum_{j=1}^m c_{ij} (y_j - y_i)$

- Electrical gap junction coupling:
  $u_i = \sum_{j=1}^m c_{ij} (y_j - y_i)$

- We assume the network graph ($G$) to be connected, weighted and undirected.
- The closed-loop system has ultimately bounded solutions.

Synchronization in networks of identical oscillators

Model Parameters: $a_i = a$, $b_i = b$, $\epsilon_i = \epsilon$, $I_i = I \forall i$

- Non-Smooth Lyapunov Analysis:
  - Sufficient condition in terms of a lower bound on the second-smallest eigenvalue of the graph Laplacian
  $\gamma \lambda_2(\mathcal{G}) > 1 + \frac{1}{3} \beta^2 + \epsilon$

- A Contraction Based Approach:
  - Yields a tighter bound
  $\gamma \lambda_2(\mathcal{G}) > 1$

Future directions

- Mixed Mode Oscillation

References

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