Synchronization in Neuronal Oscillator Networks

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Motivation

- Synchronized activity is crucial for brain function:
  - Basal ganglia
  - Local Field Potential
  - fMRI/functional connectivity

- Knowledge about conditions for synchronization can lead to a better understanding of:
  - Deep Brain Stimulation
  - Transcranial Stimulation
  - System Identification
  - Testable predictions
  - Measurable efficacy metrics for disease treatment

Linden et al. (2014), LFPy: a tool for biophysical simulation of extracellular potentials generated by detailed model neurons, Frontiers in Neuroinformatics.

MIT Tech. Review (March 2016), Halo Neuroscience
Asymptotically stable synchronization in a network of homogeneous semi-passive neuronal oscillators is guaranteed with sufficient coupling.

Changes in the stability of synchronization/consensus manifold result from:

- Graph Structure
- Coupling
- External Inputs
- Time Delay
- Oscillator properties

Neuronal Oscillator: Fitzhugh-Nagumo (FN)

- Second order dynamics for membrane potential
  - Fast Dynamics: \( \dot{y} = y - \frac{y^3}{3} - z + I \)
  - Slow Dynamics: \( \dot{z} = \epsilon(y - bz + a) \)

The dynamics of FN Oscillator model is strictly semi-passive.
- Outside a ball around the origin, a strictly semi-passive system behaves as a strictly passive system.
Network of FN Oscillators

Dynamics of a Single Neuron in the Network

- $\dot{y}_i = y_i - \frac{y_i^3}{3} - z_i + I_i + u_i$
- $\dot{z}_i = \epsilon(y_i - bz_i + a)$

Electrical gap junction coupling:

- $u_i = \sum_{j=1}^{n} \gamma_{ij} (y_j - y_i), \quad \gamma_{ij} \geq 0$

$$\Rightarrow \quad u = -\Gamma y$$

$$\Gamma = \begin{bmatrix}
\sum \gamma_{1j} & -\gamma_{12} & \cdots & -\gamma_{1n} \\
-\gamma_{21} & \sum \gamma_{2j} & \cdots & -\gamma_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-\gamma_{n1} & -\gamma_{n2} & \cdots & \sum \gamma_{nj}
\end{bmatrix}$$

The closed-loop system has ultimately bounded solutions.\(^2\)

- In finite time, solutions of the closed-loop system enter a compact set that is invariant under the system dynamics.

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A Sufficient Condition for Synchronization

- **Lyapunov Function:**
  \[ V = \frac{1}{2} \sum_{j=2}^{n} \left[ (y_j - y_1)^2 + (z_j - z_1)^2 \right] \]

- **Lyapunov Theorem** exploits bounds (arising out of semi-passivity) on solution, i.e.
  \[ |y_i| < \beta_y, \quad \forall i \in \{1, 2, \cdots, n\} \]

- Lower bound on second-smallest eigenvalue of the graph Laplacian.

\[ \lambda_2(\Gamma) > 1 + \frac{1}{3} \beta_y^2 + \frac{1}{4b} \left( \epsilon + \frac{1}{\epsilon} - 2 \right) \stackrel{\Delta}{=} \lambda_s^* \]

\[ \lambda_2(\Gamma) = 2.29 < 5.45 = \lambda_s^* \]
A Tighter Bound

- Non-smooth Lyapunov Function:
  \[ V = \max_{i,j \in \{1, \ldots, n\}} |y_i - y_j| + \max_{i,j \in \{1, \ldots, n\}} |z_i - z_j| \]

- Lower bound on second-smallest eigenvalue of the graph Laplacian.
  \[ \lambda_2(\Gamma) > 1 + \frac{1}{3} \beta_y^2 + \epsilon \triangleq \lambda_m^* \]

\[ \lambda_2(\Gamma) = 2.29 > 2.22 = \lambda_m^* \]
Input Heterogeneity in a Complete Graph

- Synchronization is only possible when the sum of external input and social influence are same across the individuals.

- Input heterogeneity gives rise to multiple clusters in the network.
  - Clusters are determined by input structure.

- A sufficient condition for synchronization of individual clusters in a complete graph, with

  \[ \gamma_{ij} = \gamma > 0, \quad i \neq j \quad \gamma_{ii} = 0 \]

\[ \gamma > \frac{3 + \beta^2_y + 3\epsilon}{3n} \]

\[ I = 0 \quad I = 1 \quad I = 2 \]
Change of Coordinates:

- **Average:**
  \[
  \xi_1 = \frac{1}{k} \sum_{j=1}^{k} y_j \quad \xi_1 = \frac{1}{k} \sum_{j=1}^{k} z_j
  \]

- **Difference from Average:**
  \[
  \xi_i = y_i - \frac{1}{k} \sum_{j=1}^{k} y_j \quad \xi_i = z_i - \frac{1}{k} \sum_{j=1}^{k} z_j \quad i \in \{2, 3, \cdots, k\}
  \]

- When the coupling is strong enough for synchronization of individual oscillators, the dynamics of the average (of membrane potential and recovery variable) becomes identical to the dynamics of a single oscillator.
The following regimes exist in this framework:

- When \( I < I_0 \), both A and B are quiescent.

- When \( I_0 < I < I_1 \)
  - and \( \gamma < \gamma_{f_0} \), A is firing and B is quiescent.
  - and \( \gamma_{f_0} < \gamma < \gamma_{f_1} \), both A and B are firing.
  - and \( \gamma > \gamma_{f_1} \), both A and B become quiescent again.

- When \( I > I_1 \)
  - and \( \gamma < \gamma_{s_0} \), A is saturated and B is quiescent.
  - and \( \gamma > \gamma_{s_0} \), both A and B are firing.
Sufficient condition for synchronization in networks of homogeneous FitzHugh-Nagumo oscillators.

Emergence of cluster synchronization due to input heterogeneity in a complete graph.
Thank You!