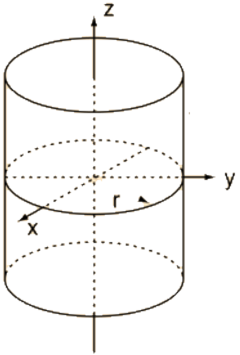
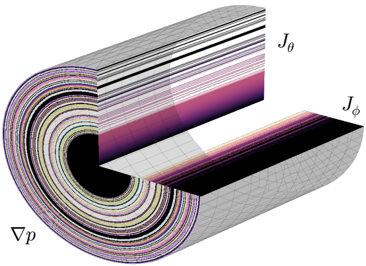
Cylindrically-Symmetric Ideal MHD Equilibria with Fractal Pressure Profiles

Brian Kraus   Stuart Hudson

October 10, 2016



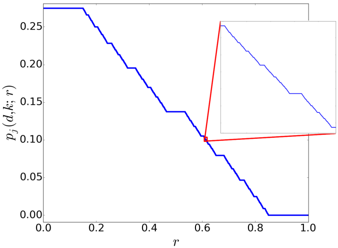


Goal: investigate physics of a fractal pressure profile

1. Why does ideal MHD require fractal pressure?

   Generally perturbed toroidal equilibria with nested flux surfaces suffer from unphysical infinite currents

   Currents vanish if the pressure is flat on all resonances



2. What mathematics will help entertain this pressure profile?

   Non-integrable fields $\implies$ KAM theory, Diophantine condition
   Dense sets, nowhere dense sets, and Lebesgue measure

3. How would a computer model this plasma profile?

   Implement a fractal grid
   Numerically quantify how robust each surface is

4. What can the cylindrically-symmetric realization of this fractal pressure tell us physically?

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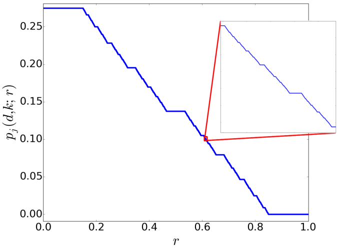

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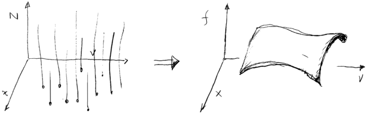
Magneto**H**ydro**D**ynamics is a simplified model for plasmas

All species' single-particle motion → single magnetic fluid



$$\xrightarrow{\text{moments}}$$

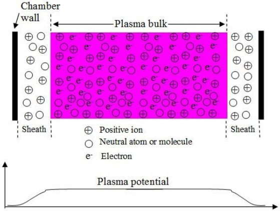
$$m\int_{-\infty}^{\infty} d^3v\; f(\boldsymbol{x},\boldsymbol{v},t) = \rho(\boldsymbol{x},t)$$

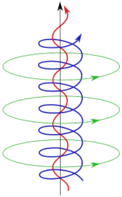
$$\frac{1}{n}\int_{-\infty}^{\infty} d^3v\; \boldsymbol{v}f(\boldsymbol{x},\boldsymbol{v},t) = \boldsymbol{u}(\boldsymbol{x},t)$$

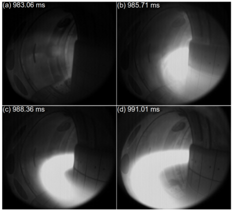
Reduces individual particle coordinates to fluid-averaged quantities

Includes variables $\quad n, \boldsymbol{u}, p, \rho = mn, \quad \boldsymbol{B}, \eta = \dfrac{1}{\sigma}, \boldsymbol{J} = \nabla\times\boldsymbol{B}/\mu_0$

Physics **not covered by** MHD:







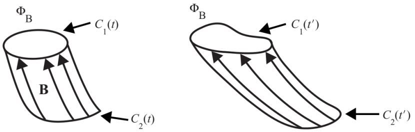
Types of MHD: Choice of Ohm's law closes model

$$\boldsymbol{J} = \underbrace{\sigma \boldsymbol{E}'}_{\text{particle frame}} = \underbrace{\sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})}_{\text{lab frame}}$$

Ideal

$\eta = 0 \rightarrow$ perfectly conductive

Flux is **frozen-in**



Magnetic topology cannot change

Resistive

$\eta = \frac{m_e \nu_{ei}}{n_e e^2} > 0$

Takes collisions into account ($\nu_{ei} \neq 0$)

Issue: ν, η scale as $T_e^{-3/2}$

At thermonuclear temperatures,

$$\eta_{\text{tokamak}} \lesssim \frac{\eta_{\text{copper}}}{10}$$

Need high numerical resolution

$$\eta_{\text{simulation}} \approx 10^4 \eta_{\text{tokamak}} \lesssim 10^{12} \eta_{\text{astro}}$$

Equilibrium states in MHD approximate macroscopic plasma structure

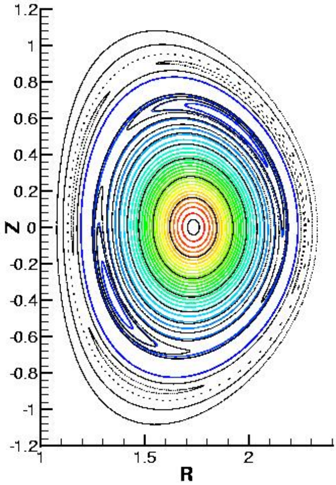


Figure: A tokamak equilibrium from NIMROD (color shows pressure).

What kinds of structure?

- Flux surfaces
- Islands
- Stochasticity / chaos

What equations lead to equilibria?

Force-balance equation

$$\rho \frac{d\boldsymbol{v}}{dt} = \boldsymbol{J} \times \boldsymbol{B} - \nabla p$$

In equilibrium:

$$d/dt \to 0 \implies \nabla p = \boldsymbol{J} \times \boldsymbol{B}$$

Three unknowns:

$$p(\boldsymbol{r}), B_T(\boldsymbol{r}), B_P(\boldsymbol{r})$$

Full solution, since $\boldsymbol{J} = \nabla \times \boldsymbol{B}/\mu_0$

How are MHD equilibria calculated?

Initial value calculations

$p(t=0)$, $\boldsymbol{B}(t=0)$

$$\rightarrow p(t \rightarrow \infty), \boldsymbol{B}(t \rightarrow \infty)$$

Goal: **evolve** states

Resistive steady-state:
M3D-C1, NIMROD

Can attack stability problems:
Sawteeth, disruptions...

Equilibrium solvers

$$p(\psi), \frac{B_T}{B_P}(\psi) \rightarrow \boldsymbol{B}(\psi), \boldsymbol{J}(\psi)$$

Get $p \rightarrow$ find matching $\boldsymbol{B}$

Straightforward in axisymmetry

Useful for finding miniumum potential state for given $p(r)$

Axisymmetry supports **nested flux surfaces**

Introduce coordinate ψ

$$B \cdot \nabla\psi = 0$$

Each surface can be treated as uniform

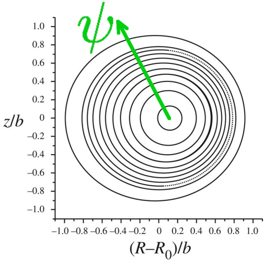
$$p(\boldsymbol{r}) = p(\psi)$$

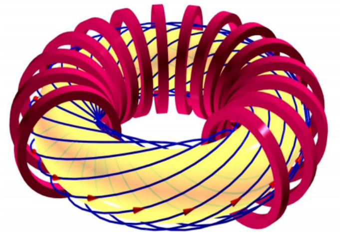
Averaged 'twist' of field lines,

$$\bar{\iota} \approx \frac{RB_P}{rB_T}$$

Measures degree of poloidal tilt

$$q = \frac{1}{\bar{\iota}} \equiv \text{safety factor}$$





Resonant magnetic perturbations modify surfaces

Axisymmetry imperfect in real systems

Fourier decompose a perturbation:

$$\xi = \sum_{m,n} \xi_{mn} \cos(m\theta - n\phi)$$

$m \equiv$ poloidal mode number;

$n \equiv$ toroidal mode number

Modes arise from:

- Error field
- Intentionally applied perturbation

Periodic ripples → field wobbles

How do nested surfaces react to field ripple?

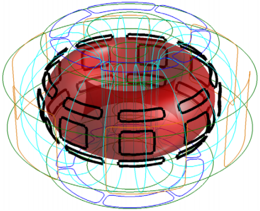
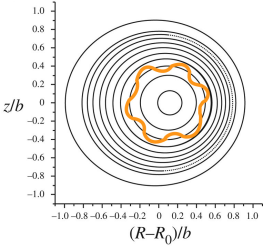


Figure: ITER's resonant magnetic coils for controlling ELMs.

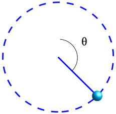


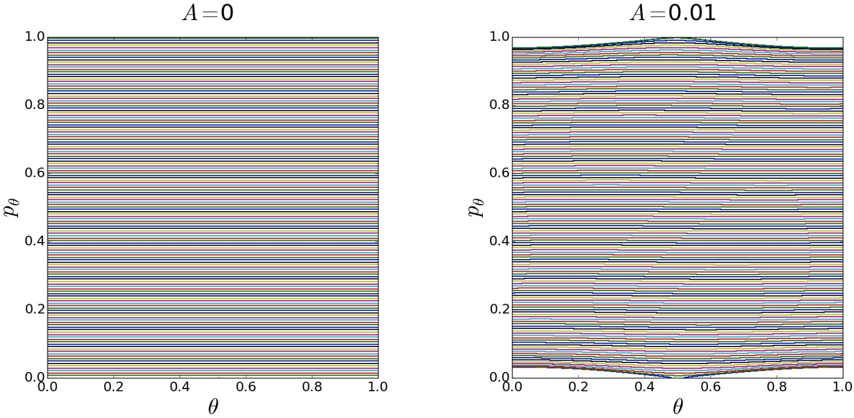
Perturbations introduce islands and chaos

The standard map models a kicked rotator

Hamiltonian with canonical coordinates (θ, p_θ):

$$p_\theta^{n+1} = p_\theta^n + A\sin(\theta^n) \qquad \theta^{n+1} = \theta^n + p_\theta^{n+1}$$





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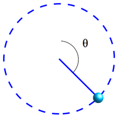

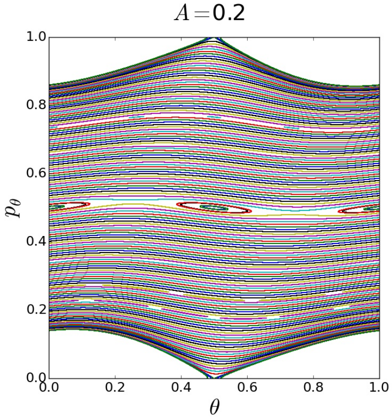

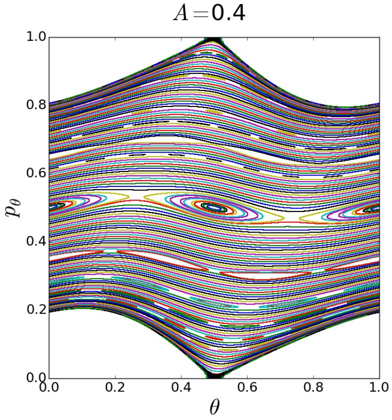

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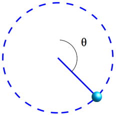

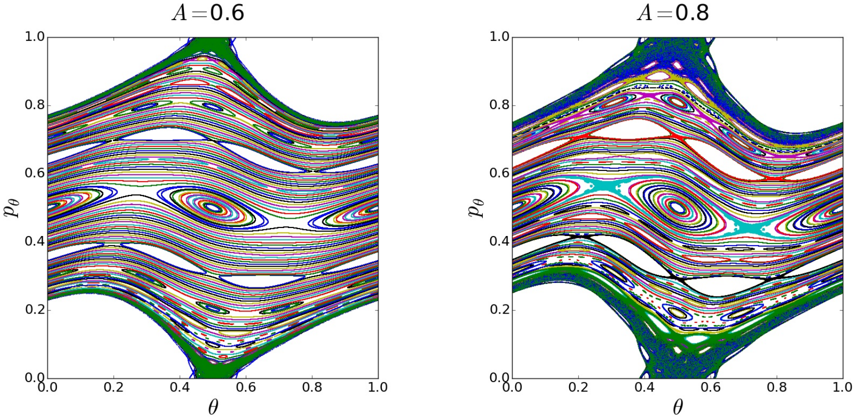

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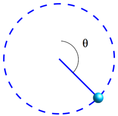

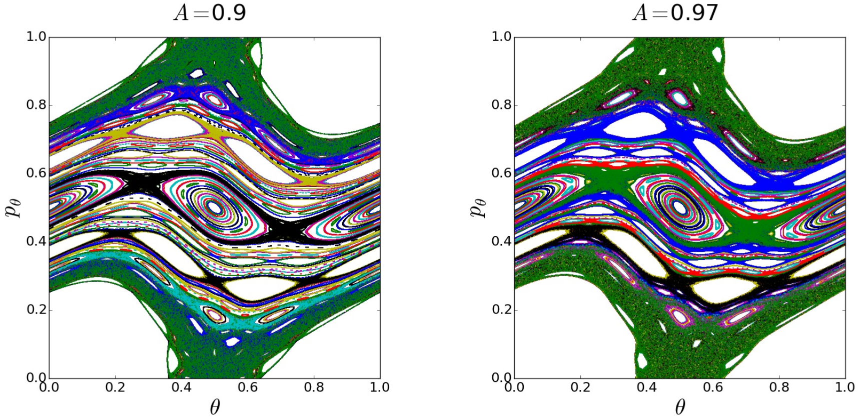

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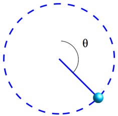
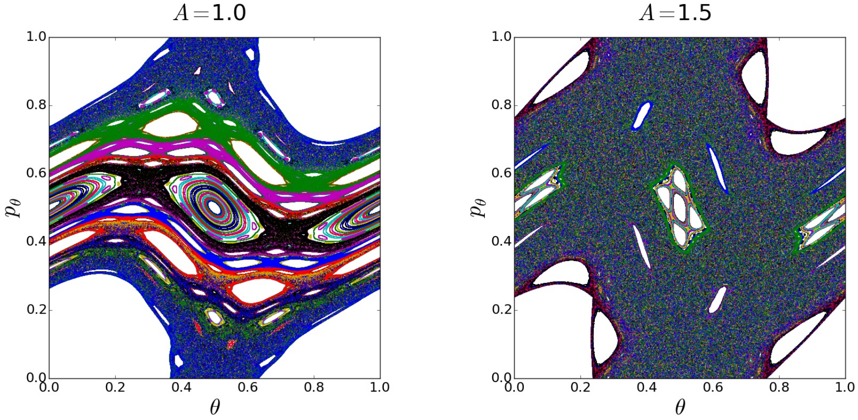




Resonant infinite currents in ideal MHD are unphysical

Begin with nested flux surfaces in equilibrium

From the equilibrium equation $\nabla p = \boldsymbol{J} \times \boldsymbol{B}$, split apart the components of $\boldsymbol{J}$:

$$\boldsymbol{J}_\perp = \frac{\boldsymbol{B} \times \nabla p}{B^2} \qquad \boldsymbol{J}_\parallel = \lambda \boldsymbol{B} \tag{1}$$

But due to particle conservation $\nabla \cdot \boldsymbol{J} = 0$:

$$\nabla \cdot \boldsymbol{J}_\perp = -\nabla \cdot (\lambda \boldsymbol{B}) = -\boldsymbol{B} \cdot \nabla \lambda \tag{2}$$

Anything of the form $\boldsymbol{B} \cdot \nabla f$ is a *magnetic differential equation*, which has a handy property in straight-field-line coordinates:

$$(\boldsymbol{B} \cdot \nabla)_{mn} = \partial_\theta \iota\!\!\!- + \partial_\phi = m \iota\!\!\!- - n \tag{3}$$

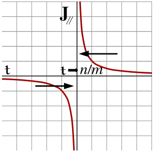
Solve for λ:

$$(\boldsymbol{B} \cdot \nabla)\lambda = -\nabla \cdot \boldsymbol{J}_\perp \tag{4}$$

Resonant infinite currents in ideal MHD are unphysical

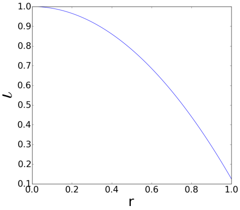
Writing $\lambda = \sum_{m,n} \lambda_{mn} \cos(m\theta - n\phi)$:

$$\lambda_{mn} = \underbrace{\Delta_{mn}\delta(m\iota\!\!\!- - n)}_{\text{1. Delta spike}} - \underbrace{\frac{(\nabla \cdot \boldsymbol{J}_\perp)_{mn}}{m\iota\!\!\!- - n}}_{\text{2. } 1/x \text{ singularity}}$$



1. Finite current through infinitesimal wire: $J = I/a \to J = I\delta(x)$ ☺
2. $\int_0^\epsilon \frac{1}{x} dx$ is logarithmically divergent → infinite current ☹

If $\iota\!\!\!- = \frac{n}{m}$, a pressure gradient drives **infinite parallel current**



$$\lambda_{mn} = \frac{(\nabla \cdot \boldsymbol{J}_\perp)_{mn}}{m\iota\!\!\!- - n} \propto \left(\nabla \cdot \frac{(\boldsymbol{B} \times \nabla p)}{B^2}\right)_{mn} \to \pm\infty$$

Unless $\nabla p = 0$ when $\iota\!\!\!- = \frac{n}{m}$

Solution: set pressure gradient to zero on all rational surfaces

Problem with $1/x$ singularity extends to **nearby intervals**

$$\int_{\epsilon}^{\delta} \frac{dx}{x} \text{ unbounded as } \epsilon \to 0$$

So, not sufficient that $\nabla p = 0$ only when $\iota = \frac{n}{m}$:

$$\text{Need to flatten } \nabla p \text{ on a neighborhood } \left(\frac{n}{m} - \delta, \frac{n}{m} + \delta\right)$$

How to pick size $\delta > 0$ for each rational resonance?

KAM theory and the Diophantine condition

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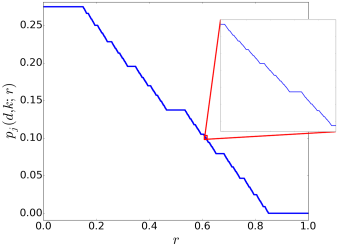

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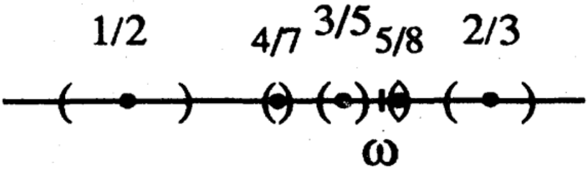
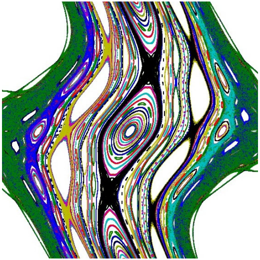
The Diophantine condition gives a criteria for where $\nabla p \neq 0$





Kolmogorov-Arnold-Moser (KAM) theory, est. 1954-1963

Small perturbations don't destroy "sufficiently irrational" surfaces

$$\text{Diophantine set } \mathcal{D}_{d,k} = \left\{ \alpha : \quad \left| \alpha - \frac{n}{m} \right| > \frac{d}{m^k} \quad , \forall n, m \in \mathbb{N} \right\}$$

As m increases, rational resonance is less pronounced

d scales with amplitude of perturbation at all resonances

k determines decay rate of perturbation as order m increases

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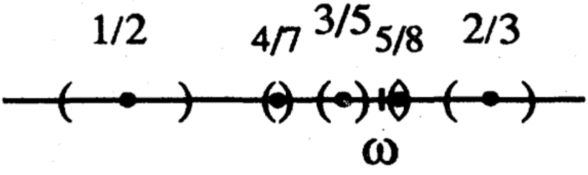

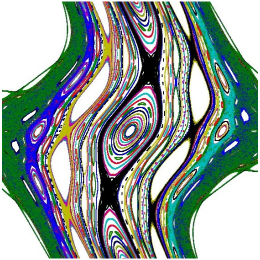

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For small $d > 0$ and $k \geq 2$, there exists $\alpha \in \mathcal{D}_{d,k}$.

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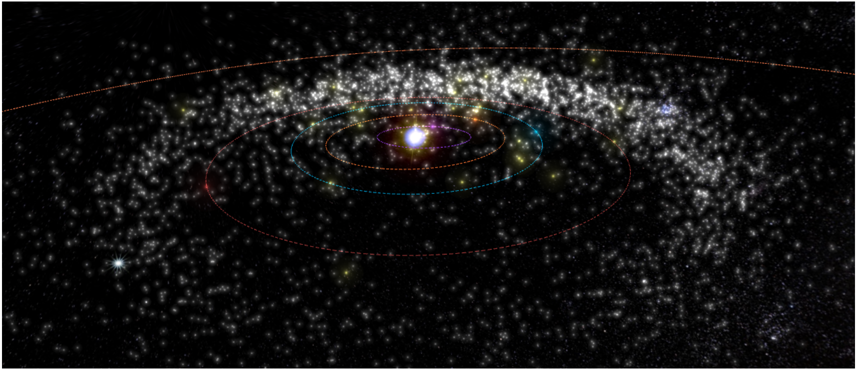
Stable orbits of asteroids show KAM in action

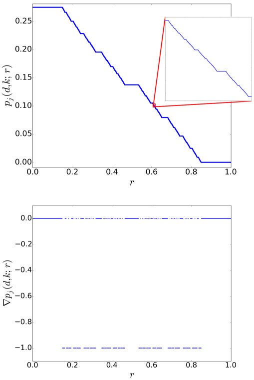
3D asteroid visualization



Realizing a pressure profile that zeroes the current singularities

So that $\nabla p = 0$ on all the *excluded rational intervals*,

$$p'(\iota\!\!\!-) = \frac{dp}{d\iota\!\!\!-} = \begin{cases} 0 & \left|\iota\!\!\!- - \frac{n}{m}\right| < \frac{d}{m^k},\ \forall\ \frac{n}{m} \in \mathfrak{F}_j; \\ -1 & \text{otherwise,} \end{cases}$$



Physical properties of such a plasma:

What is β?

$$\int_0^1 dr\ p(r) \implies \text{Lebesgue integration}$$

How do we prove $p(r) \neq 0, \forall r$?

Where is $\nabla p = 0$ distributed in $\iota\!\!\!-$?

Staircase $p(r) \implies$ what fields $\boldsymbol{B}, \boldsymbol{J}$?

Numerically: Can we approximate this on a discrete grid?

Grad visualized this decades ago

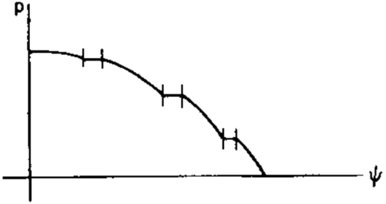
THE PHYSICS OF FLUIDS VOLUME 10, NUMBER 1 JANUARY 1967

Toroidal Containment of a Plasma

Harold Grad
Courant Institute of Mathematical Sciences, New York University, New York, New York
(Received 5 July 1966; final manuscript received 10 October 1966)

The question of plasma containment in a torus is much more complicated than in an open-ended mirror system. Serious questions arise of the nonexistence of flux surfaces, of noncontained particle drifts, and of nonexistence of self-consistent equilibria at small gyroradius.

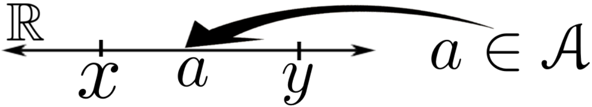
Fig. 5. Pressure function.



"The function p is continuous but its derivative is pathological. We have obtained an equilibrium solution without infinite currents, but at the price of a very pathological pressure distribution."

A **dense** set does not necessarily have **measure**

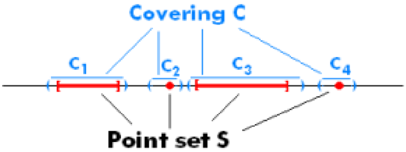
Dense: for $x, y \in \mathcal{A}$, there is always an $a \in \mathcal{A}$ with $x < a < y$



Rationals $\mathbb{Q}$, irrationals $\mathbb{I}$ are both dense

But, measure $\mu(\mathbb{Q}) = 0$, while $\mathbb{I}$ has full measure

$$\mu(\mathcal{A}) \equiv \text{minimum open-interval covering of } \mathcal{A}$$



Difference is **countability**: $\mu(\text{countable \# discrete points}) = 0;$

$\mu(\text{uncountable \# discrete points}) = \cdots$

The Cantor set $\mathcal{C}$ is uncountable, but has no measure

Remove middle third from every interval

Remaining points are **nowhere dense** on the real line

$$\mu(\mathcal{C}) = 1 - \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}}$$
$$= 1 - \frac{1}{3}\left(\frac{1}{1-2/3}\right) = 1 - 1 = 0$$

Lost intervals are *countable* $\implies$ $\mathcal{C}$ is uncountable!

Our $\nabla p \neq 0$ surfaces are topologically the same as fat Cantor

If removed fraction is less than 1/3, then any measure is possible

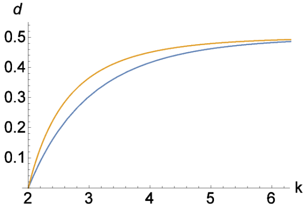
$$0 \leq \mu(\mathcal{C}_*) < 1$$

Fat Cantor's removed fraction $\sim$ Diophantine parameters (d, k)

Sum of rational interval widths,

$$\sum_{m,n} \frac{2d}{m^k},$$

is less than 1 for d shown vs. k



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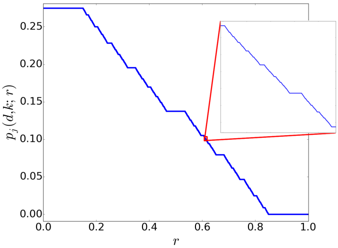

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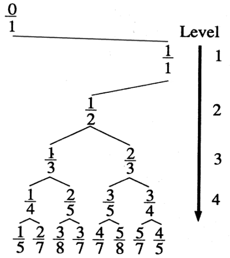
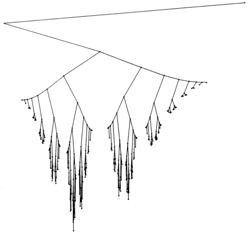
Numerics based on the Diophantine condition use a **fractal grid**





Farey tree sorts rationals n/m

Discretization problem: Farey tree is fractal

> A **fractal** object exhibits **self-similarity** at all scales.

Most numerical methods don't apply

Solution: Build fractal structure into numerical grid, one Farey level at a time

$$x_0 = [0, 1]$$

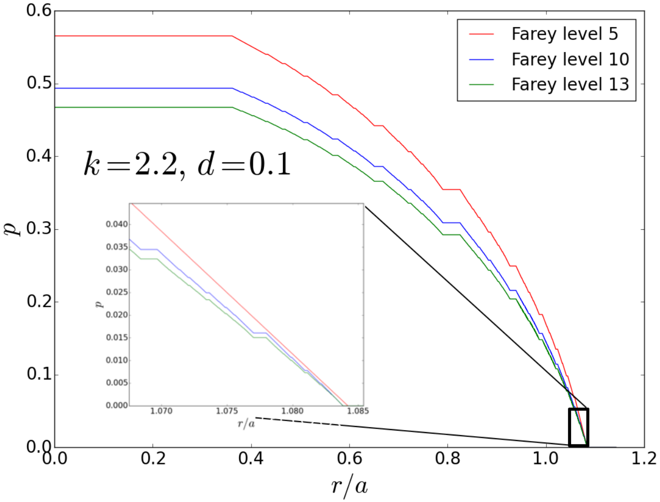
$$x_1 = \left[0, \quad \frac{d}{1^k}, \quad 1 - \frac{d}{1^k}, \quad 1\right]$$

$$\vdots$$

$$x_i = \left[0, \text{boundaries of all rational regions}, 1\right]$$

More Farey tree levels → better approximation to true fractal

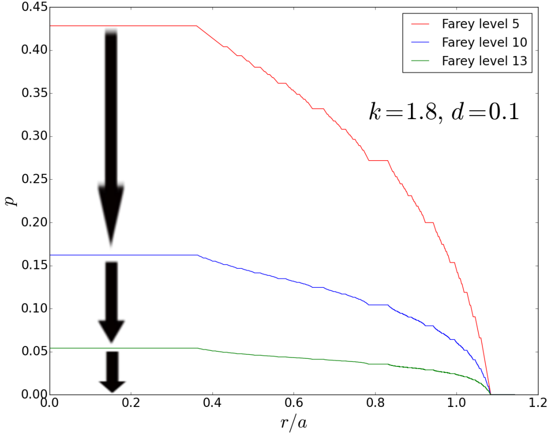
$$p'(t) = \begin{cases} 0 & \left|t - \frac{n}{m}\right| < \frac{d}{m^k},\ \forall\ \frac{n}{m} \in \mathcal{F}_j \\ -1 & \text{otherwise,} \end{cases}, \quad p(r) = \int_0^r p'(r')dr', \quad t = 1 - \frac{7r^2}{8}$$



Numerical approximations are not accurate beyond the bounds of number-theoretical analysis

Finite-spaced grid approximations $p_{\Delta x}$ should converge as $\Delta x \to 0$

$$\lim_{\Delta x \to 0} (p_{\Delta x} - p_{\mathrm{fractal}}) = 0$$

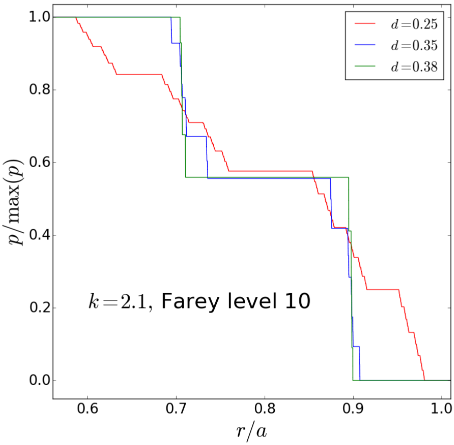


More resolution brings $p(r) \to 0$

Analysis shows $\mu(\mathcal{D}_{d,k}) > 0$ only for $k > 2$

The most irrational surfaces are most robust to perturbation

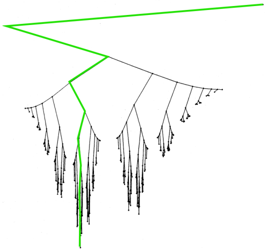
Noble numbers are 'furthest' from rationals



Most noble number:

$$\frac{1}{\varphi} = \frac{\sqrt{5}-1}{2}$$

$\iota = \frac{1}{\varphi}$: last remaining ∇p

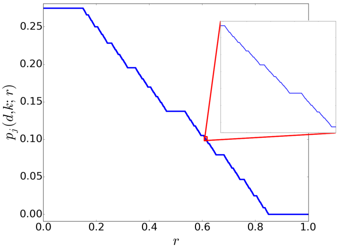


Goal: investigate physics of a fractal pressure profile

1. Why does ideal MHD require fractal pressure?

   Generally perturbed toroidal equilibria with nested flux surfaces suffer from unphysical infinite currents

   Currents vanish if the pressure is flat on all resonances



2. What mathematics will help entertain this pressure profile?

   Non-integrable fields $\implies$ KAM theory, Diophantine condition
   Dense sets, nowhere dense sets, and Lebesgue measure

3. How would a computer model this plasma profile?

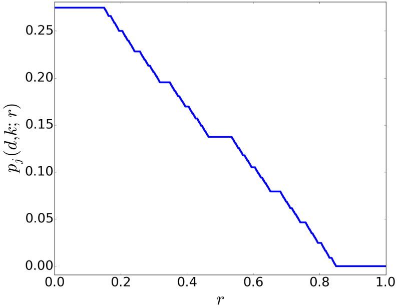
   Implement a fractal grid
   Numerically quantify how robust each surface is

4. What can the cylindrically-symmetric realization of this fractal pressure tell us physically?

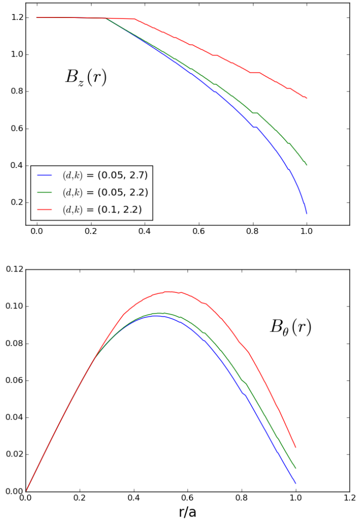
Fractal behavior of $p(r)$ carries over to $\boldsymbol{B}$

Fractal pressure and $\boldsymbol{B}$ are **continuous**, but not **smooth**

Smoothness broken on a nowhere-dense set of finite measure



$\Longrightarrow$

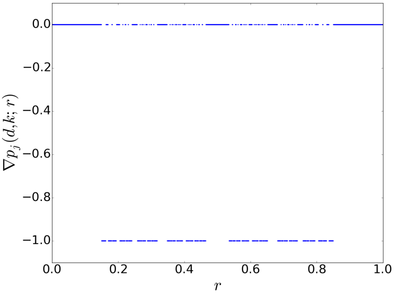


Discontinuous currents have different profiles on rational and irrational subsets

$\boldsymbol{J}$ has two behaviors:

1. Force-free profile on rational intervals
2. Discontinuities on nowhere-dense irrational subset

But, $\boldsymbol{J}$ is always finite



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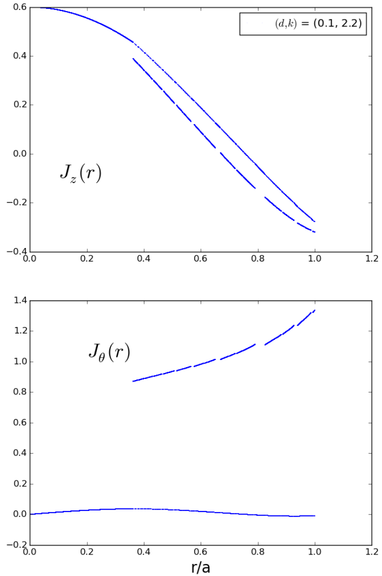

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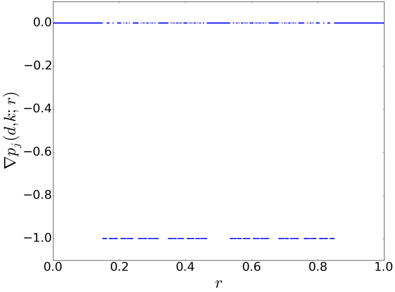

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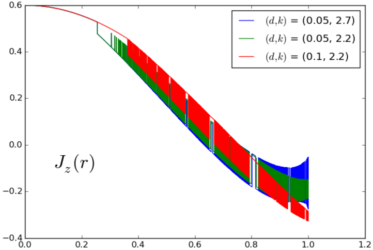

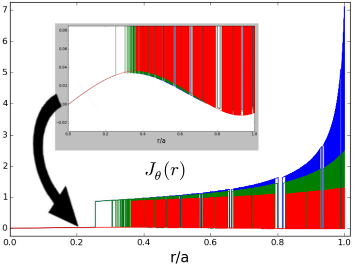

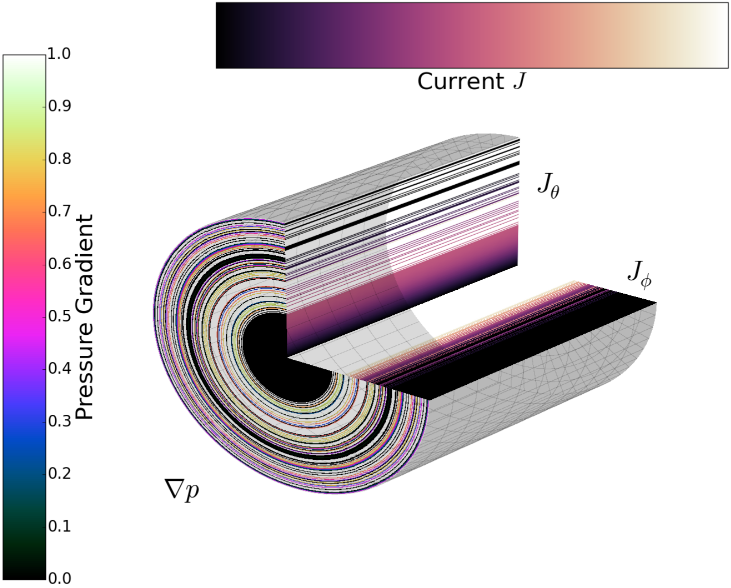
Current J
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
Pressure Gradient
J_θ
J_ϕ
∇p

Conclusion

Grad's "pathological" solution to current singularities in MHD is a valid equilibrium state

Number theory and measure theory guarantee that a nontrivial fractal pressure can exist close to axisymmetry

Ascertaining the number-theoretical commonalities of the most robust irrationals is underway

Fractal pressure is compatible with non-smooth $\boldsymbol{B}(\boldsymbol{r})$ and discontinuous (but finite!) $\boldsymbol{J}(\boldsymbol{r})$