# ELE539A: Optimization of Communication Systems
# Lecture 19: Interior Point Algorithms for Constrained Convex Optimization

Professor M. Chiang
Electrical Engineering Department, Princeton University

March 29, 2006

## Lecture Outline

- Inequality constrained minimization problems
- Barrier function and central path
- Barrier method

## Inequality Constrained Minimization

Let $f_0, \ldots, f_m : \mathbf{R}^n \to \mathbf{R}$ be convex and twice continuously differentiable and $A \in \mathbf{R}^{p \times n}$ with rank $p < n$:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

Assume the problem is strictly feasible and an optimal $x^*$ exists

Idea: reduce it to a sequence of linear equality constrained problems and apply Newton's method

First, need to approximately formulate inequality constrained problem as an equality constrained problem

# Barrier Function

Make inequality constraints implicit in the objective:

$$\begin{array}{ll} \text{minimize} & f_0(x) + \sum_{i=1}^m I_-(f_i(x)) \\ \text{subject to} & Ax = b \end{array}$$

where $I_-$ is indicator function:

$$I_-(u) = \begin{cases} 0 & u \leq 0 \\ \infty & u > 0 \end{cases}$$

No inequality constraints, but objective function not differentiable

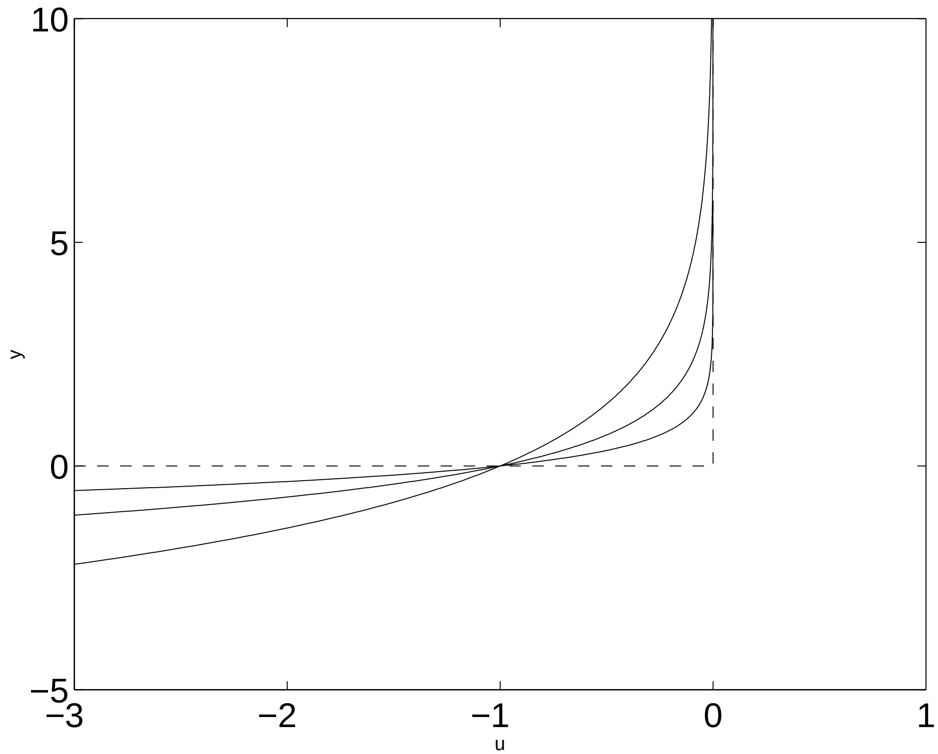
Approximate indicator function by a differentiable, closed, and convex function:

$$\hat{I}_-(u) = -(1/t)\log(-u), \quad \mathbf{dom}\,\hat{I}_- = -\mathbf{R}_{++}$$

where a larger parameter $t$ gives more accurate approximation

$\hat{I}_-$ increases to $\infty$ as $u$ increases to $0$

# Log Barrier




Use Newton method to solve approximation:

$$\begin{array}{ll} \text{minimize} & f_0(x) + \sum_{i=1}^{m} \hat{I}_-(f_i(x)) \\ \text{subject to} & Ax = b \end{array}$$

# Log Barrier

Log barrier function:

$$\phi(x) = -\sum_{i=1}^{m} \log(-f_i(x)), \quad \mathbf{dom}\,\phi = \{x \in \mathbf{R}^n | f_i(x) < 0, i = 1, \ldots, m\}$$

Approximation better if $t$ is large, but then Hessian of $f_0 + (1/t)\phi$ varies rapidly near boundary of feasible set. Accuracy Stability tradeoff

Solve a sequence of approximation with larger $t$, using Newton method for each step of the sequence

Gradient and Hessian of log barrier function:

$$\begin{aligned}
\nabla\phi(x) &= \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla f_i(x) \\
\nabla^2\phi(x) &= \sum_{i=1}^{m} \frac{1}{f_i(x)^2} \nabla f_i(x) \nabla f_i(x)^T + \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla^2 f_i(x)
\end{aligned}$$

## Central Path

Consider the family of optimization problems parameterized by $t > 0$:

$$\begin{array}{ll} \text{minimize} & tf_0(x) + \phi(x) \\ \text{subject to} & Ax = b \end{array}$$

Central path: solutions to above problem $x^*(t)$, characterized by:

1. Strict feasibility:

$$Ax^*(t) = b, \quad f_i(x^*(t)) < 0, \;\; i = 1, \ldots, m$$

2. Centrality condition: there exists $\hat{\nu} \in \mathbf{R}^p$ such that

$$t\nabla f_0(x^*(t)) + \sum_{i=1}^{m} \frac{1}{-f_i(x^*(t))} \nabla f_i(x^*(t)) + A^T \hat{\nu} = 0$$

Every central point gives a dual feasible point. Let

$$\lambda_i^*(t) = -\frac{1}{tf_i(x^*(t))}, \;\; i = 1, \ldots, m \quad \nu^*(t) = \frac{\hat{\nu}}{t}$$

# Central Path

Dual function

$$g(\lambda^*(t), \nu^*(t)) = f_0(x^*(t)) - m/t$$

which implies duality gap is $m/t$. Therefore, suboptimality gap

$$f_0(x^*(t)) - p^* \leq m/t$$

Interpretation as modified KKT condition:

$x$ is a central point $x^*(t)$ iff there exits $\lambda, \nu$ such that

$$\begin{aligned}
Ax = b, \;\; f_i(x) \leq 0, & \qquad i = 1, \ldots, m \\
\lambda \succeq 0 & \\
\nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + A^T \nu = 0 & \\
-\lambda_i f_i(x) = 1/t, & \qquad i = 1, \ldots, m
\end{aligned}$$

Complementary slackness is relaxed from $0$ to $1/t$

## Example

Inequality form LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b \end{array}$$

Log barrier function:

$$\phi(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x)$$

with gradient and Hessian:

$$\begin{aligned} \nabla\phi(x) &= \sum_{i=1}^{m} \frac{a_i}{b_i - a_i^T x} = A^T d \\ \nabla^2\phi(x) &= \sum_{i=1}^{m} \frac{a_i a_i^T}{(b_i - a_i^T x)^2} = A^T \,\mathbf{diag}(d^2) A \end{aligned}$$

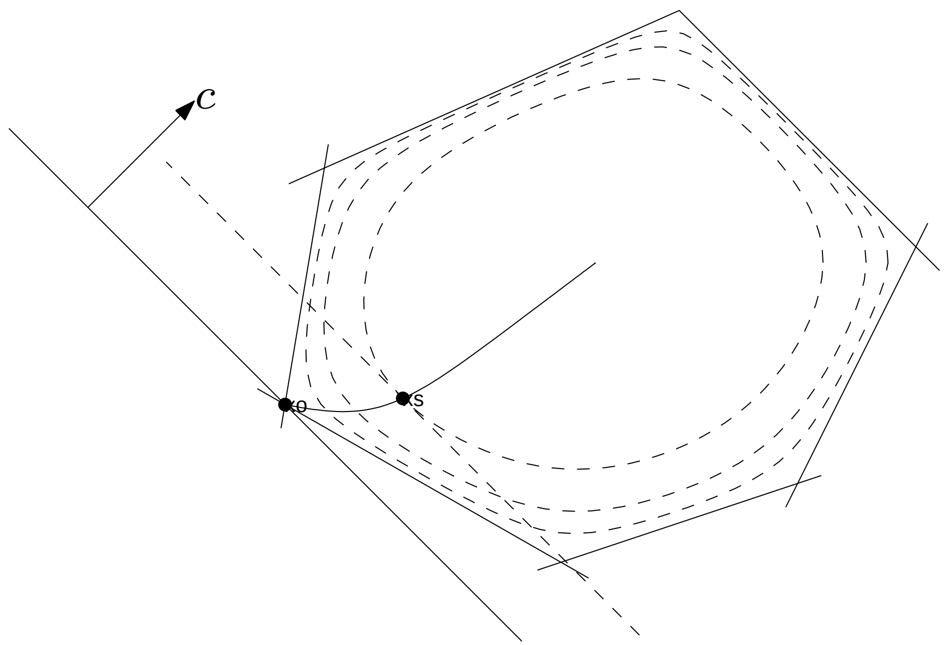
where $d_i = 1/(b_i - a_i^T x)$

Centrality condition becomes:

$$tc + A^T d = 0$$

$c$ is parallel to $\nabla\phi(x)$.

Therefore, hyperplane $c^T x^*(t)$ is tangent to level set of $\phi$



## Barrier Method

GIVEN a strictly feasible point $x \in \mathbf{dom}\, f, t := t^{(0)} > 0, \mu > 1$ and tolerance $\epsilon > 0$

REPEAT

1. Centering step: compute $x^*(t)$ by minimizing $tf_0 + \phi$ subject to $Ax = b$, starting at $x$

2. Update: $x := x^*(t)$

3. Stopping criterion: QUIT if $\frac{m}{t} \leq \epsilon$

4. Increase $t$: $t := \mu t$

Other names: Sequential Unconstrained Minimization Technique (SUMT) or path-following method

Usually use Newton method for Centering Step

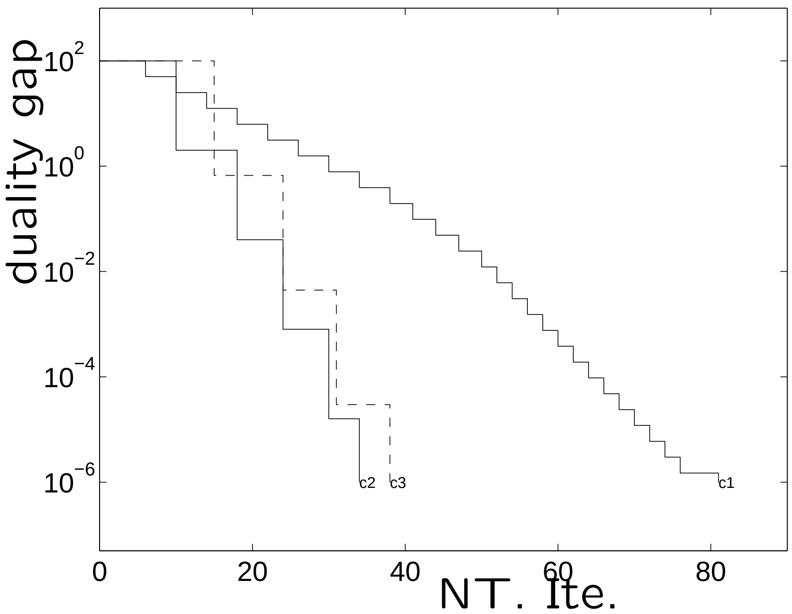
## Remarks

- Each centering step does not need to be exact
- Choice of $\mu$: tradeoff number of inner iterations with number of outer iterations
- Choice of $t^{(0)}$: tradeoff number of inner iterations within the first outer iteration with number of outer iterations
- Number of centering steps required:

$$\frac{\log(m/(\epsilon t^{(0)}))}{\log \mu}$$

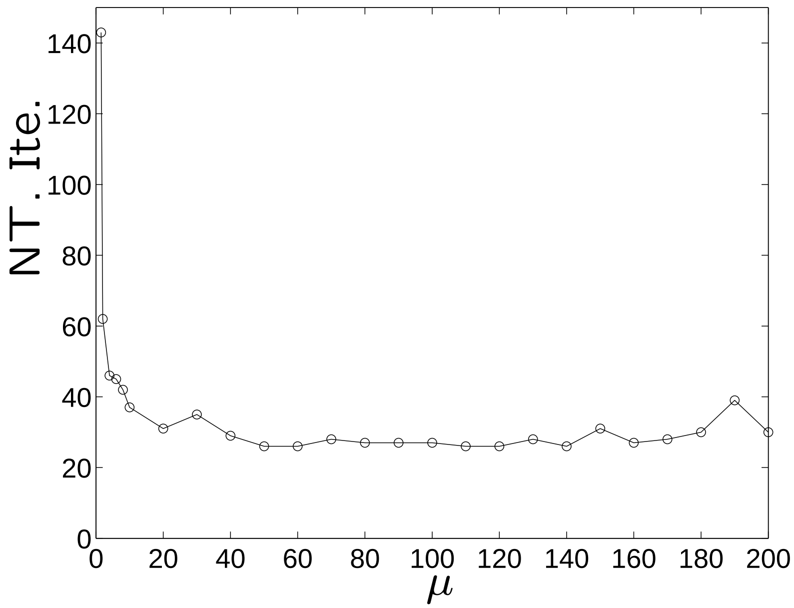
where $m$ is the number of inequality constraints and $\epsilon$ is desired accuracy

## Progress of Barrier Method for an LP Example

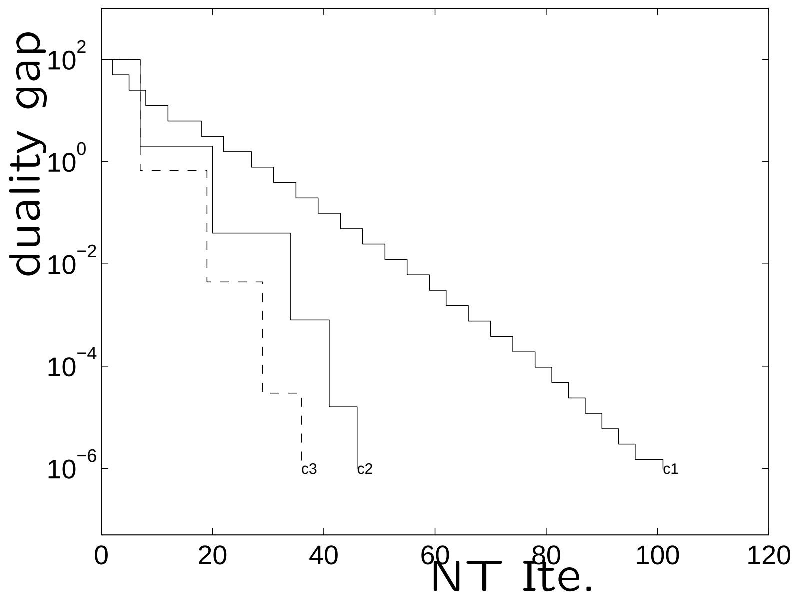



Three curves for $\mu = 2, 50, 150$

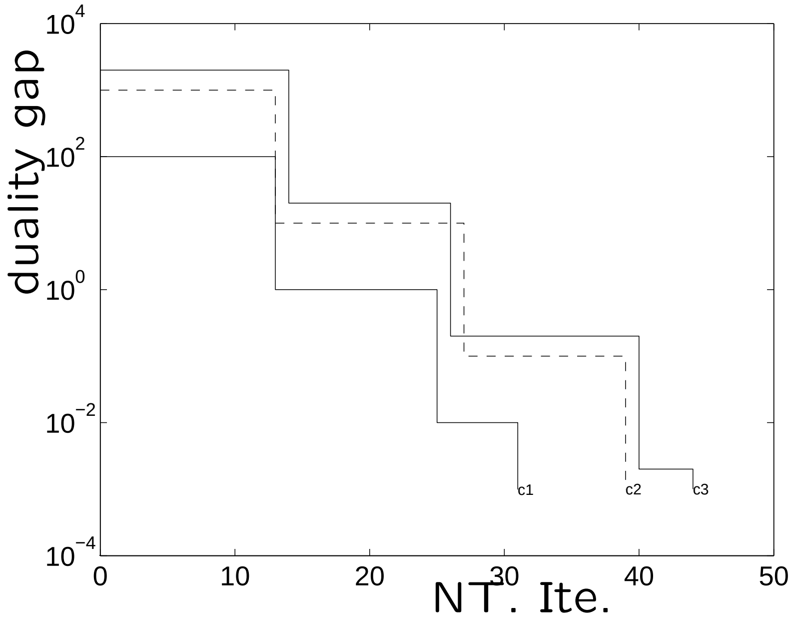
# Tradeoff of $\mu$ Parameter for a Small LP

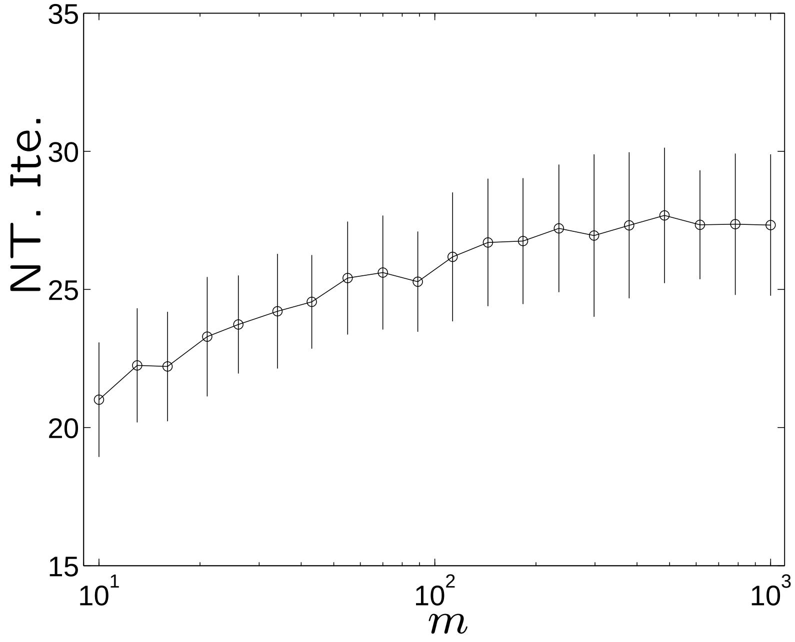


## Progress of Barrier Method for a GP Example



## Insensitive to Problem Size




Three curves for $m = 50, 500, 1000$, $n = 2m$.

# Phase I Method

How to compute a strictly feasible point to start barrier method?

Consider a phase I optimization problem in variables $x \in \mathbf{R}^n, s \in \mathbf{R}$:

$$
\begin{array}{ll}
\text{minimize} & s \\
\text{subject to} & f_i(x) \leq s, \quad i = 1, \dots, m \\
& Ax = b
\end{array}
$$

Strictly feasible point: for any $x^{(0)}$, let $s = \max f_i(x^{(0)})$

1. Apply barrier method to solve phase I problem (stop when $s < 0$)

2. Use the resulted strictly feasible point for the original problem to start barrier method for the original problem

## Not Covered

- Self-concordance analysis and complexity analysis (polynomial time)
- Numerical linear algebra (large scale implementation)
- Generalized inequalities (SDP)

Other (sometimes more efficient) algorithms:

- Primal-dual interior point method
- Ellipsoid methods
- Analytic center cutting plane methods

## Lecture Summary

- Solve a general convex optimization with interior point methods
- Turn inequality constrained problem into a sequence of equality constrained problems that are increasingly accurate approximation of the original problem
- Polynomial time (in theory) and much faster (in practice): about 25-50 least-squares effort for a wide range of problem sizes

Readings: Chapter 11.1-11.4 in Boyd and Vandenberghe