Optimization of Communication Systems
Power Control in Cellular Networks

Mung Chiang
Electrical Engineering Department, Princeton University

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Overview

Part	*Objective*	*Constraints*	*Mode*	*PHY*
I	Energy efficiency		Game	Multi-carrier
II	Concave utility	Feasible region	Optimization	Single carrier
III	General	Individual QoS	Optimization	Single carrier

Acknowledgement: Tian Lan, Farhad Meshkati, Prashanth Hande, Sundeep Rangan, Chee Wei Tan, Daniel Palomar

References

- F. Meshkati, M. Chiang, H. V. Poor, and S. Schwartz, "A game-theoretic approach to energy-efficient power control in multi-carrier CDMA systems", *IEEE JSAC*, 2006.

- P. Hande, S. Rangan, and M. Chiang, "Distributed uplink power control for optimal SIR assignment in cellular data networks", *Proc. IEEE INFOCOM*, 2006.

- M. Chiang, C. W. Tan, D. Palomar, D. O'Neill, and D. Julian, "Power control by geometric programming", To appear in *IEEE Trans. Wireless Comm.*, 2007.

Part I

Multi-Carrier Energy-Efficiency Power Control Game

Related Work

- MacKenzie and Wicker 2001
- Xiao, Shroff and Chong 2001
- Alpcan and Altman 2001
- Saraydar, Mandayam, and Goodman 2002
- Yu, Ginis, and Cioffi 2002
- Sung and Wong 2003

Open problems:

- Energy efficiency as utility function $\Rightarrow$ Non-quasiconcave utilities
- Multiple carriers $\Rightarrow$ Vector strategy

Energy Efficiency Utility Function

l: carrier index. D carriers

k: user index. K users

$\gamma_{kl} = \frac{p_{kl} h_{kl}}{\eta + \frac{1}{N} \sum_{j \neq k} p_{jl} h_{jl}}$: SIR for user k on carrier l

$f(\gamma_{kl})$: reliability function (sigmoidal function)

Throughput: $T_{kl} = R_k f(\gamma_{kl})$

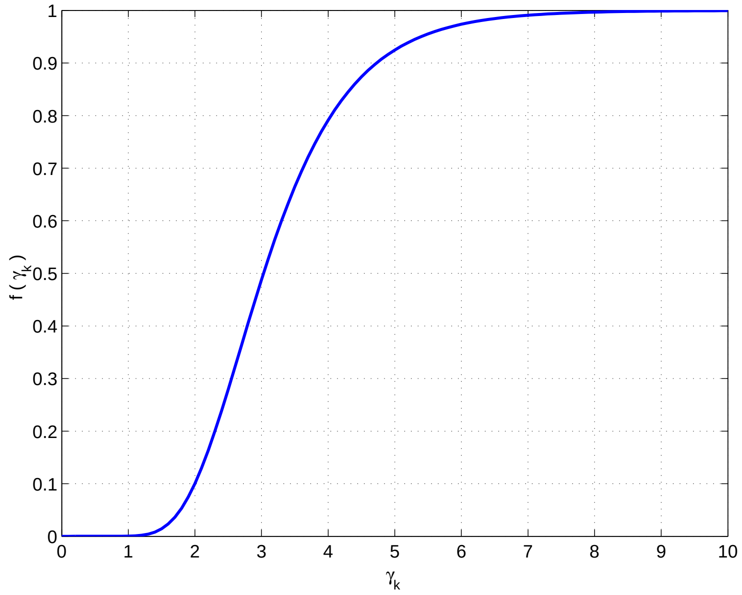
Power: p_{kl}

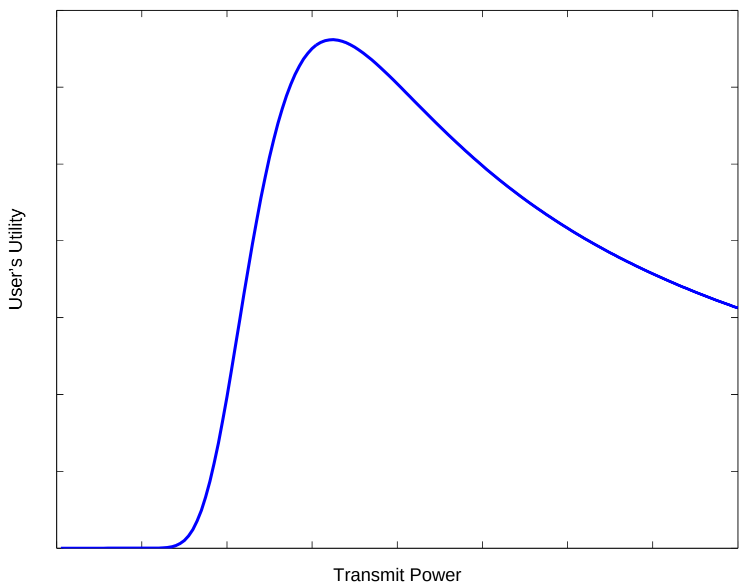
Energy efficiency utility function: $u_k = \frac{\sum_{l=1}^{D} T_{kl}}{\sum_{l=1}^{D} p_{kl}}$

Local and selfish utility maximization: $\max_{\mathbf{p}_k} u_k$

Game: $[\{1, 2, \ldots, K\}, \{[0, P_{max}]_k^D\}, \{u_k\}]$

Reliability Function and Energy Efficiency Utility





Multi-Carrier Energy Efficiency Maximization

γ^*: unique positive solution of $f(\gamma) = \gamma f'(\gamma)$

p^*_{kl}: transmit power needed to achieve SIR γ^* (or P_{max} if γ^* is not attainable)

L_k: $\mathrm{argmin}_l \, p^*_{kl}$ ('best' carrier)

Theorem: Energy efficiency maximizer is $p_{kl} = p^*_{kL_k}$ for $l = L_k$ and 0 otherwise

Only transmit on the 'best' carrier

Reduces number of possibilities of NE to D^K

Characterization of NE

Channel gains $\{h_{jl}\}$ determine NE possibilities

First assume that γ^* is attainable by all users (large enough processing gain N)

Define $\Theta_n = \frac{1}{1-(n-1)\frac{\gamma^*}{N}}, \;\; n = 0, 1, \ldots, K$
$(0 < \Theta_0 < \Theta_1 = 1 < \Theta_2 < \ldots < \Theta_K)$

$n(i)$: number of users transmitting on carrier i

Theorem: For user k to transmit on carrier l at NE:

$$\frac{h_{kl}}{h_{ki}} > \frac{\Theta_{n(l)}}{\Theta_{n(i)}} \Theta_0, \;\; \forall i \neq l$$

and in this case, $p^*_{kl} = \gamma^* \sigma^2 \frac{\Theta_{n(l)}}{h_{kl}}$

Existence and Uniqueness of NE

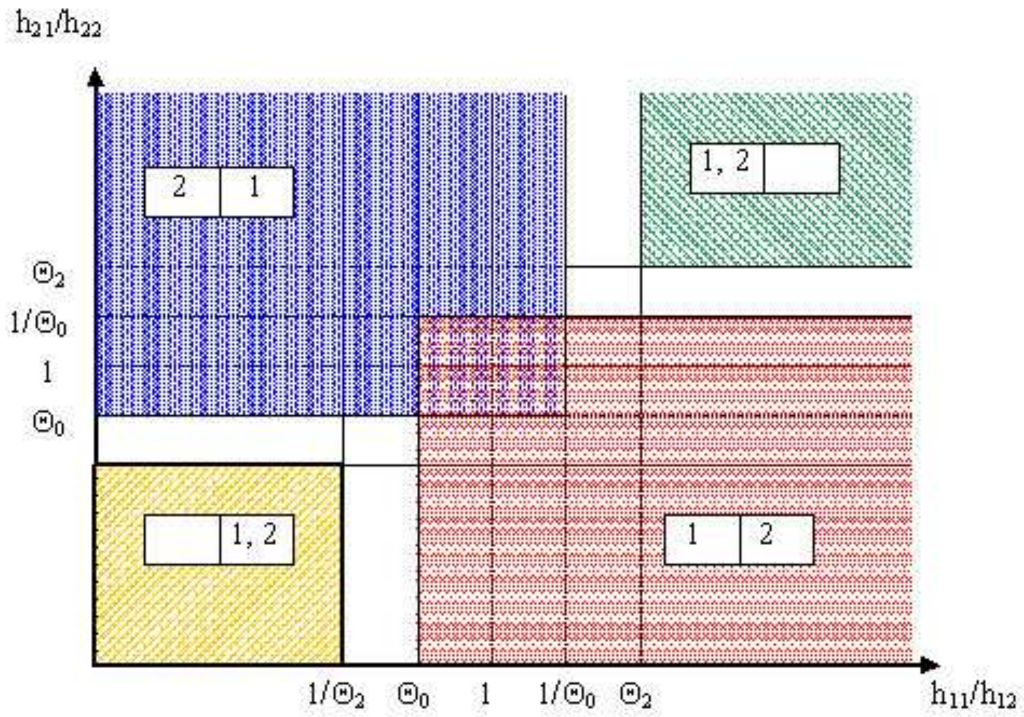
Existence of NE: Sufficient condition is that channel gains satisfy $K(D-1)$ inequalities simultaneously

Uniqueness of NE: Not guaranteed

$(K=2, D=2)$ case. Four possibilities:

- $(1,2|)$: $\frac{h_{11}}{h_{12}} > \Theta_2$ and $\frac{h_{21}}{h_{22}} > \Theta_2$
- $(|1,2)$: $\frac{h_{11}}{h_{12}} < \frac{1}{\Theta_2}$ and $\frac{h_{21}}{h_{22}} < \frac{1}{\Theta_2}$
- $(1|2)$: $\frac{h_{11}}{h_{12}} > \Theta_0$ and $\frac{h_{21}}{h_{22}} < \frac{1}{\Theta_0}$
- $(2|1)$: $\frac{h_{11}}{h_{12}} < \frac{1}{\Theta_0}$ and $\frac{h_{21}}{h_{22}} > \Theta_0$

Example



Homogeneity of channel gains: If either h_{11}/h_{22} or h_{22}/h_{11} belongs to $[1/\Theta_2^2, \Theta_0^2]$, then there does not exist NE

Two-Carrier Two-User Case

Rayleigh fading channel: $h_{kl} = \frac{c}{d_k^{-4}} a_{kl}^2$

a_{kl}: i.i.d. and have Rayleigh distribution with mean 1

X_1: number of users transmitting over first carrier at NE

$$P_{X_1}(0) = P_{X_1}(2) = \begin{cases} 0 & \text{if } N \leq \gamma^* \\ \left(\frac{1}{1+\Theta_2}\right)^2 & \text{if } N > \gamma^* \end{cases},$$

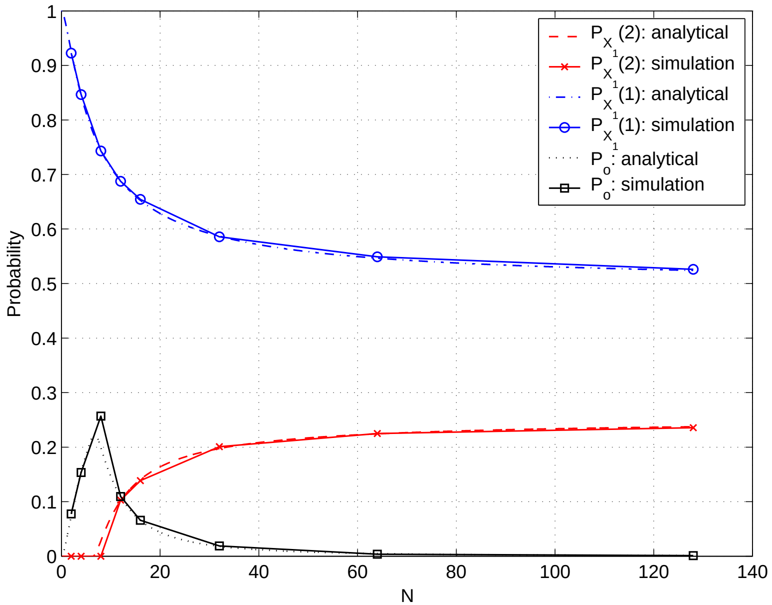
$$P_{X_1}(1) = 2\left(\frac{1}{1+\Theta_0}\right)^2 - \left(\frac{1-\Theta_0}{1+\Theta_0}\right)^2,$$

$$P_o = \begin{cases} 2\left(\frac{\Theta_0}{1+\Theta_0}\right)^2 & \text{if } N \leq \gamma^* \\ 2\left[\left(\frac{\Theta_0}{1+\Theta_0}\right)^2 - \left(\frac{1}{1+\Theta_2}\right)^2\right] & \text{if } N > \gamma^* \end{cases}.$$

Two-Carrier Case

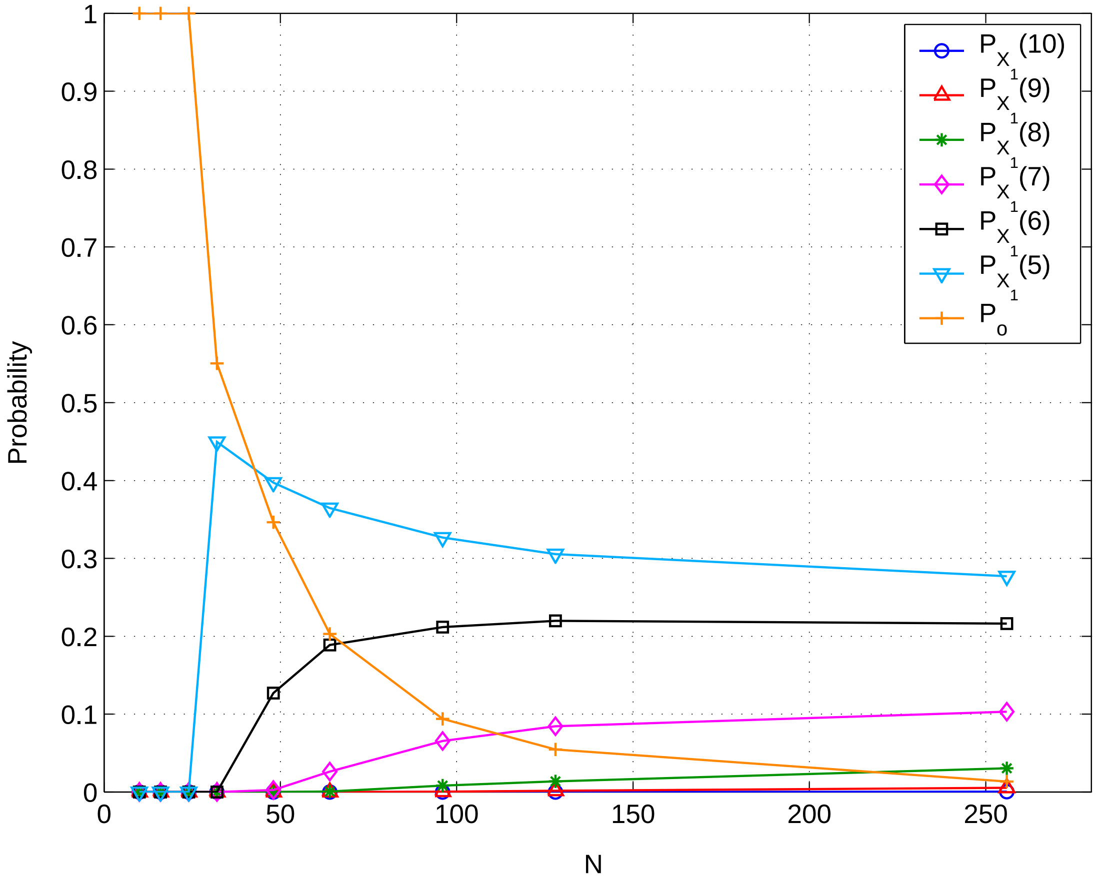
As processing gain N becomes large, there exists a unique NE well approximated by:

$$Pr\{X_1 = m\} \approx C_m^K (0.5)^K, \;\; m = 0, 1, \ldots, K$$



Example (2 carriers, 10 users)

No NE when N is too small, and always exists NE as $N \to \infty$



Distributed Algorithm

Best response, iterative, distributed

Algorithm:

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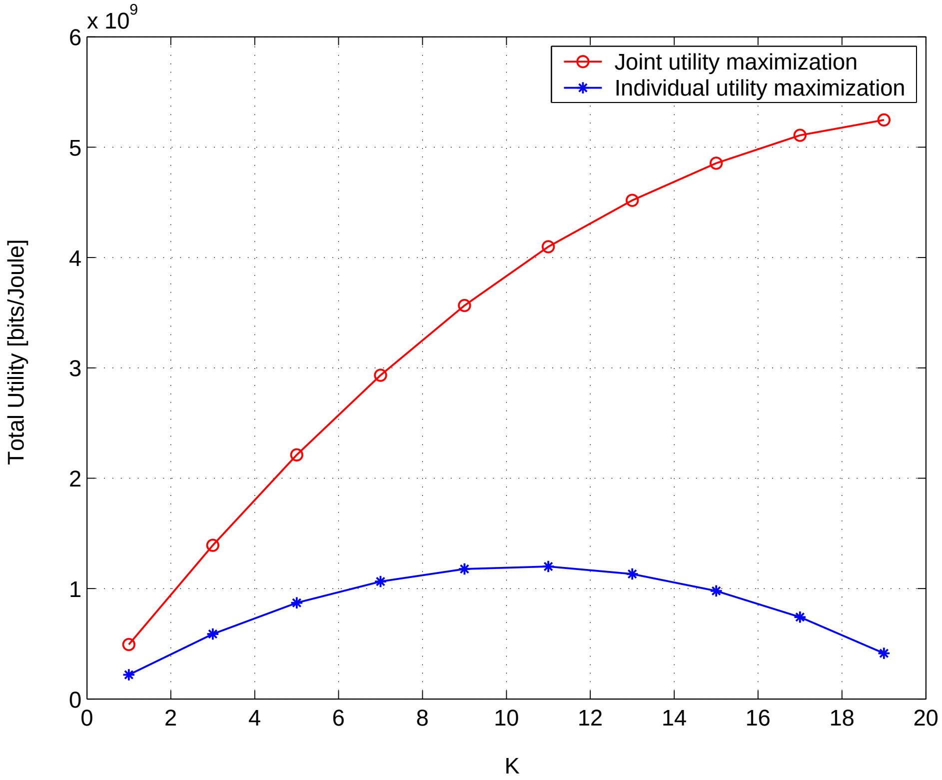
1. $k = 1$
2. User k picks "best" carrier and transmits on it only, at power level to attain SIR γ^*
3. $k = (k+1)$ mod K

Theorem: Above algorithm converges to NE (when it exists) for all two-user and three-user cases

Observation: Always converges to NE in all cases

Performance Gain

Comparison between our vector-valued strategy game and optimization over individual carriers



Part II

Distributed Jointly Optimal SIR Assignment and Power Control

Related Work

Uplink power control in multi-cellular networks

• Fixed SIR: distributed power control:

Zander 1992, Foschini Miljanic 1993, Mitra 1993, Yates 1995, Bambos Pottie 2000

• Nash equilibrium for joint SIR assignment and power control:

Saraydar, Mandayam, and Goodman 2001, 2002

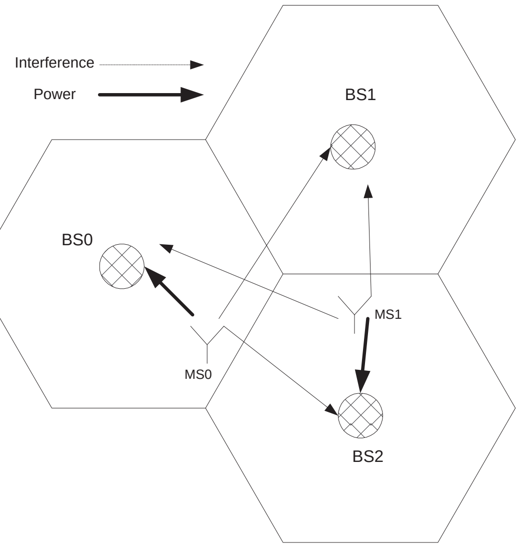
• Centralized computation for globally optimal joint SIR assignment and power control:

Chiang 2004, O'Neill, Julian, and Boyd 2004, Boche and Stanczak 2004

Open Problem: Distributed and optimal joint SIR assignment and power control

Solved by Hande, Rangan, and Chiang 2006

System Model



M MS and N BS

Each MS i served by a BS σ_i

Each BS k serving a set of MS: S_k

C_i: set of interference links

- Non-orthogonal system: $C_i = \{j \mid j \neq i\}$
- Orthogonal system: $C_i = \{j \mid \sigma_j \neq \sigma_i\}$

Notation

h^0_{kj}: absolute path gain from MS j to BS k

$h_{kj} = h^0_{kj}/h^0_{\sigma_j j}$: normalized path gain

$M \times M$ matrix $\mathbf{G}$: path gain matrix

$$G_{ij} = \begin{cases} h_{\sigma_i j} & \text{if } j \in C_i, \\ 0 & \text{if } j \notin C_i, \end{cases}$$

p_j: received signal power on link j at its serving BS σ_j

q_j: total interference and noise at the BS serving MS j

η_j: noise on link j

γ_j: SIR achieved by link j. $\gamma_j = p_j/q_j$

$$\begin{aligned} \mathbf{q} &= \mathbf{G}\mathbf{p} + \boldsymbol{\eta} = \mathbf{G}\mathbf{D}(\boldsymbol{\gamma})\mathbf{q} + \boldsymbol{\eta} \\ \mathbf{p} &= \mathbf{D}(\boldsymbol{\gamma})\mathbf{G}\mathbf{p} + \mathbf{D}(\boldsymbol{\gamma})\boldsymbol{\eta} \end{aligned}$$

Review: Feasibility

Not all SIR assignments are feasible

$\boldsymbol{\gamma} \succ 0$ is feasible if there exists an interference vector, $\mathbf{q} \succeq 0$, and power vector $\mathbf{p} \succeq 0$, satisfying basic equations

Lemma (Zander 1992):

If $\boldsymbol{\eta} = 0$, $\boldsymbol{\gamma} \succ 0$ is feasible iff $\rho(\mathbf{GD}(\boldsymbol{\gamma})) = 1$. Given a feasible $\boldsymbol{\gamma}$, $\mathbf{p}(\boldsymbol{\gamma})$ is the right Perron-Frobenious eigenvector of $\mathbf{GD}(\boldsymbol{\gamma})$ and $\mathbf{q}(\boldsymbol{\gamma})$ is the right Perron-Frobenious eigenvector of $\mathbf{D}(\boldsymbol{\gamma})\mathbf{G}$

If $\boldsymbol{\eta} \neq 0$, $\boldsymbol{\gamma} \succ 0$ is feasible iff $\rho(\mathbf{GD}(\boldsymbol{\gamma})) < 1$. Given a feasible $\boldsymbol{\gamma}$,

$$
\begin{aligned}
\mathbf{p}(\boldsymbol{\gamma}) &= (\mathbf{I} - \mathbf{D}(\boldsymbol{\gamma})\mathbf{G})^{-1}\mathbf{D}(\boldsymbol{\gamma})\boldsymbol{\eta} \\
\mathbf{q}(\boldsymbol{\gamma}) &= (\mathbf{I} - \mathbf{GD}(\boldsymbol{\gamma}))^{-1}\boldsymbol{\eta}
\end{aligned}
$$

Feasible Regions

Assume $\boldsymbol{\eta} \neq 0$, feasible region:

$$\mathbf{B} = \{\boldsymbol{\gamma} \succ 0 : \rho(\mathbf{GD}(\boldsymbol{\gamma})) < 1\}$$

Finite power case: given $\rho \in [0, 1)$

$$\mathbf{B}_\rho = \{\boldsymbol{\gamma} \succ 0 \mid \rho(\mathbf{GD}(\boldsymbol{\gamma})) \leq \rho\}$$

Power constrained case:

$$\mathbf{B}(\mathbf{p}^m) = \{\boldsymbol{\gamma} \in \mathbf{B} \mid \mathbf{p}(\boldsymbol{\gamma}) \preceq \mathbf{p}^m\}$$

Interference constrained case:

$$\mathbf{B}(\mathbf{q}^m) = \{\boldsymbol{\gamma} \in \mathbf{B} \mid \mathbf{q}(\boldsymbol{\gamma}) \preceq \mathbf{q}^m\}$$

Pareto-Optimal Boundary

Theorem on Pareto-Optimal Boundary:

	Feasible Region	Pareto-optimal Boundary
1	$\mathbf{B} = \{\boldsymbol{\gamma} : \rho(\mathbf{GD}(\boldsymbol{\gamma})) < 1\}$	$\partial\mathbf{B} = \{\boldsymbol{\gamma} : \rho(\mathbf{GD}(\boldsymbol{\gamma})) = 1\}$
2	$\mathbf{B}_\rho = \{\boldsymbol{\gamma} : \rho(\mathbf{GD}(\boldsymbol{\gamma})) < \rho\}$	$\partial\mathbf{B}_\rho = \{\boldsymbol{\gamma} : \rho(\mathbf{GD}(\boldsymbol{\gamma})) = \rho\}$
3	$\mathbf{B}(\mathbf{p}^m) = \{\boldsymbol{\gamma} : \rho(\mathbf{GD}(\boldsymbol{\gamma})) < 1, \mathbf{p} \preceq \mathbf{p}^m\}$	$\partial\mathbf{B}(\mathbf{p}^m) = \{\boldsymbol{\gamma} : \rho < 1, \mathbf{p} \preceq \mathbf{p}^m, \exists i : p_i = p_i^m\}$
4	$\mathbf{B}(\mathbf{q}^m) = \{\boldsymbol{\gamma} : \rho(\mathbf{GD}(\boldsymbol{\gamma})) < 1, \mathbf{p} \preceq \mathbf{q}^m\}$	$\partial\mathbf{B}(\mathbf{q}^m) = \{\boldsymbol{\gamma} : \rho < 1, \mathbf{q} \preceq \mathbf{q}^m, \exists i : q_i = q_i^m\}$

Load-Spillage Characterization

Lemma: $\boldsymbol{\gamma} \succ 0$ is feasible (and ρ-optimal) iff there exists a $\mathbf{s} \succ 0$ and $\rho \in [0, 1)$ such that

$$\mathbf{s}^T \mathbf{G}\mathbf{D}(\boldsymbol{\gamma}) = \rho \mathbf{s}^T$$

Let $\mathbf{r}(\mathbf{s}) = \mathbf{G}^T \mathbf{s}$

A new view on SIR: $\boldsymbol{\gamma}(\mathbf{s}, \rho) = \rho \mathbf{s}/\mathbf{r}(\mathbf{s})$

$\mathbf{s}$ and $\mathbf{r}$ are left eigenvectors of the matrices, $\mathbf{D}(\boldsymbol{\gamma})\mathbf{G}$ and $\mathbf{G}\mathbf{D}(\boldsymbol{\gamma})$ (corresponding to eigenvalues ρ)

$\mathbf{s}$: Load vector: $s_i = r_i \gamma_i / \rho$

$\mathbf{r}$: Spillage vector: $r_i = \sum_j G_{ji} s_j$

Alternative to power-interference characterization

Key to distributed algorithm

Attaining Pareto-Optimality

Algorithm:

Initialize: Fixed $\mathbf{s} \succ 0$ and $\rho \in [0, 1)$.

1. BS k broadcasts the BS-load factor $\ell_k = \sum_{j \in S_k} s_j$.
2. Compute the spillage factor $r_i = \sum_{j \neq i, j \in S_{\sigma_i}} s_j + \sum_{k \neq \sigma_i} h_{ki} \ell_k$.
3. Assign SIR values $\gamma_i = \rho s_i / r_i$.

Stop. The resulting SIR vector $\boldsymbol{\gamma} = \boldsymbol{\gamma}(\mathbf{s}, \rho)$.

Alternative versions: MS-Control or BS-Control

Power and Interference Constrained Cases

Introduce power price vector $\boldsymbol{\nu}$

Algorithm:

Initialize: fixed $s_i[0] \succ 0, \nu_i[0] \succeq 0$.

1. BS k broadcasts the BS-load factor $\ell_k[t] = \sum_{j \in S_k} s_j[t]$.
2. Compute the spillage factor
   $r_i[t] = \sum_{j \neq i, j \in S_{\sigma_i}} s_j[t] + \sum_{k \neq \sigma_i} h_{ki} \ell_k[t] + \nu_i$.
3. Assign SIR $\gamma_i[t] = s_i / r_i[t]$ for MS i: .
4. Measure resulting power $p_i[t]$.
5. Update power price $\nu_i[t+1] = [\nu_i[t] + \delta(p_i[t] - p_i^m)]^+$.

Continue: $t := t + 1$.

Utility Maximization

Which Pareto-optimal $\boldsymbol{\gamma}$ to pick?

Maximize concave utility functions over Pareto-optimal boundary

Utility functions $U(\boldsymbol{\gamma}) = \sum_i U_i(\gamma_i)$:

- Strictly increasing, twice differentiable with bounded derivatives, strictly concave in $\log \gamma_i$
- No starvation: As $\gamma_i \to 0$, $U_i(\gamma_i) \to -\infty$

Intuition: Assign higher SIR to

- MS with good channel condition (power-interference view)
- MS with worse interfering channel condition (load-spillage view)

Distributed Algorithm

Algorithm:

Initialize: Arbitrary $\mathbf{s}[0] \succ 0$.

1. BS k broadcasts the BS-load factor $\ell_k[t] = \sum_{i \in S_k} s_i[t]$.
2. Compute the spillage-factor $r_i[t]$ by $\sum_{j \neq i, j \in S_{\sigma_i}} s_j + \sum_{k \neq \sigma_i} h_{ki} \ell_k$.
3. Assign SIR values $\gamma_i[t] = s_i[t]/r_i[t]$.
4. Measure the resulting interference $q_i[t]$.
5. Update the load factor $s_i[t]$:

$$s_i[t+1] = s_i[t] + \delta \Delta s_i[t].$$

where $\Delta s_i = \frac{U_i'(\gamma_i)\gamma_i}{q_i} - s_i$

Continue: $t := t + 1$.

Convergence and Optimality

Theorem: For sufficiently small step size $\delta > 0$, Algorithm converges to the globally optimal solution of

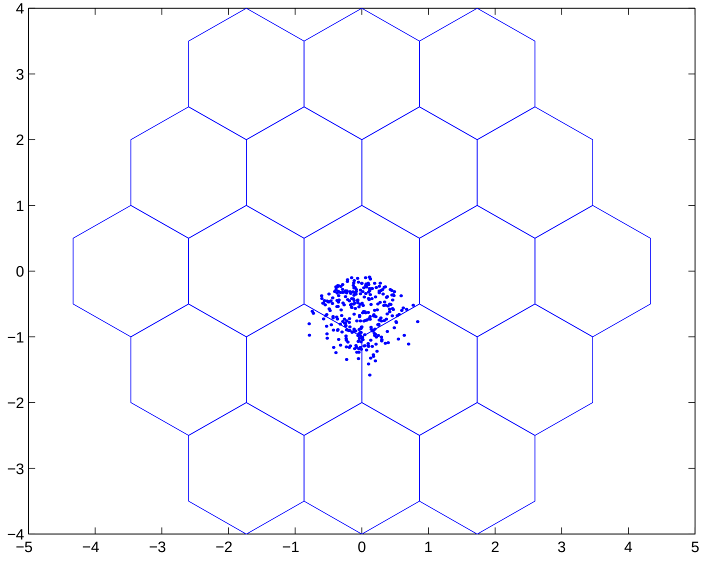
$$\begin{array}{ll} \text{maximize} & U(\boldsymbol{\gamma}) \\ \text{subject to} & \rho(\mathbf{D}(\boldsymbol{\gamma})\mathbf{G}) \leq 1 \end{array}$$

Proof: Key ideas:

- Develop ascent direction (most involved)
- Evaluate KKT conditions
- Guarantee Lipschitz condition

Extend to power and interference constrained cases

Simulation



3GPP Uplink Evaluation Tool: 19 cells in three hexagons

Each cell divided into three 120 degree sectors, 57 base station sectors

Uniform distribution of MS

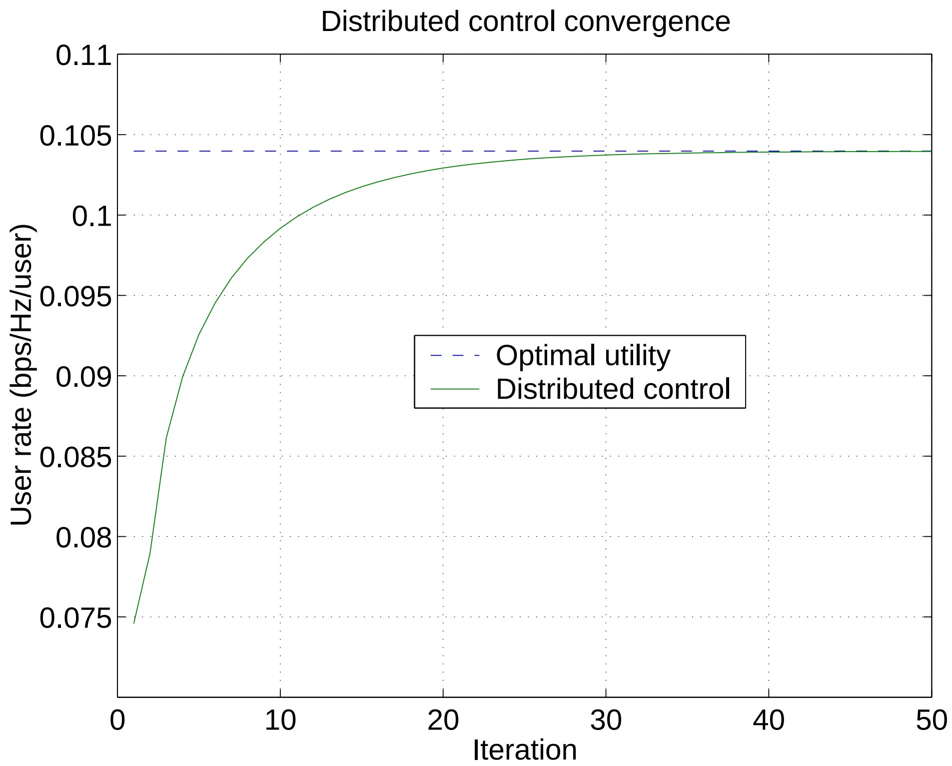
Antenna: 65 degree 3 dB bandwidth, 15 dB antenna gain

Channel: Pass loss exponent: 3.7, log-normal shadowing: 8.9 dB

Convergence

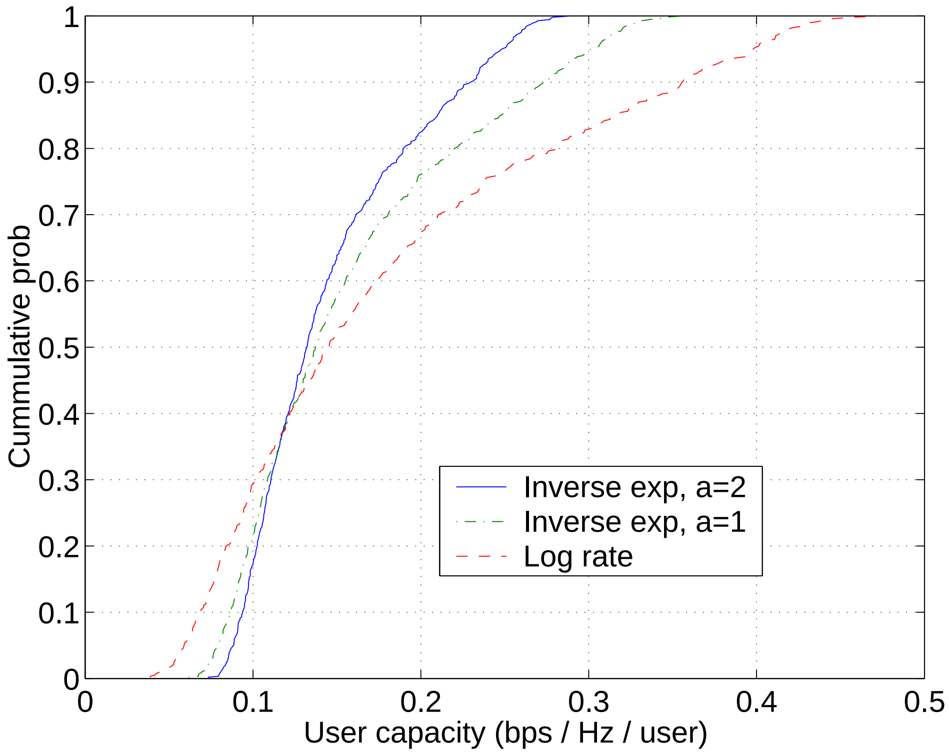
10 MS per sector, 570 MS in total

Fast convergence with distributed control



Utility

Effects of shapes of utility function



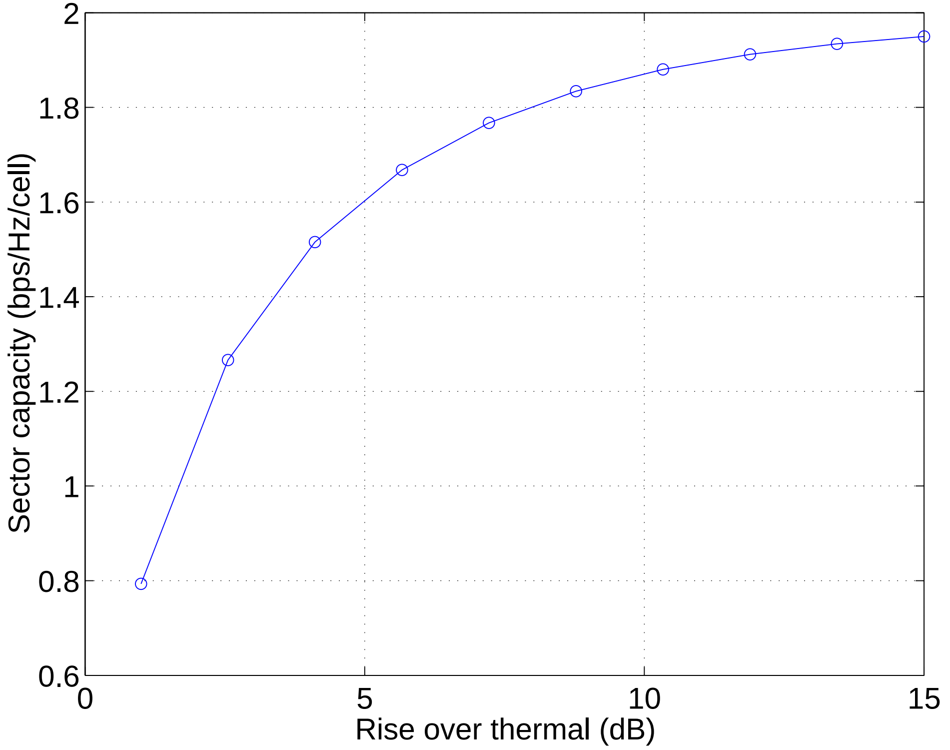
Sector Capacity and Fairness

Tradeoff between efficiency and fairness

Utility function	Sector capacity (bps / Hz / sector)	10% Worst User capacity (bps / Hz)
Log QoS	1.90	0.055
Inv exp, $a = 1$	1.58	0.086
Inv exp, $a = 2$	1.46	0.094
Inv exp, $a = 3$	1.46	0.097

Spectral Efficiency and MS Power Consumption

Interference-limited version of distributed algorithm
Tradeoff between sector capacity and Rise-Over-Thermal limit



Part III

Nonconvex Formulations of Constrained Power Control Optimization

Related Work

- O'Neill, Chiang, Julian, and S. Boyd 2002
- Kandukuri and Boyd 2002
- Boche and Stanczak 2004
- Papandriopoulos, Dey, and Evans 2005
- Chiang 2005

Convexity and SIR Regime

Constrained optimization formulations (more general than concave utility maximization over convex feasible region):

Objective: Network-wide metric (*e.g.*, system throughput, min max fairness)

Constraints: Individual user QoS requirements (*e.g.*, rate, delay, outage)

SIR regime matters:

- High SIR: pseudo-nonconvexity
- Medium to Low SIR: real nonconvexity

Notation

Signal Interference Ratio:

$$\mathsf{SIR}_i(\mathbf{P}) = \frac{P_i h_{ii}}{\sum_{j \neq i}^{N} P_j h_{ij} + \eta_i}.$$

Attainable data rate at high SIR:

$$c_i(\mathbf{P}) = \frac{1}{T} \log_2(1 + K\mathsf{SIR}_i(\mathbf{P})).$$

Outage probability on a wireless link:

$$P_{o,i}(\mathbf{P}) = \mathbf{Prob}\{\mathsf{SIR}_i(\mathbf{P}) \leq \mathsf{SIR}_{th}\}$$

Average (Markovian) queuing delay with Poisson(Λ_i) arrival:

$$\bar{D}_i(\mathbf{P}) = \frac{1}{c_i(\mathbf{P}) - \Lambda_i}$$

GP Formulations

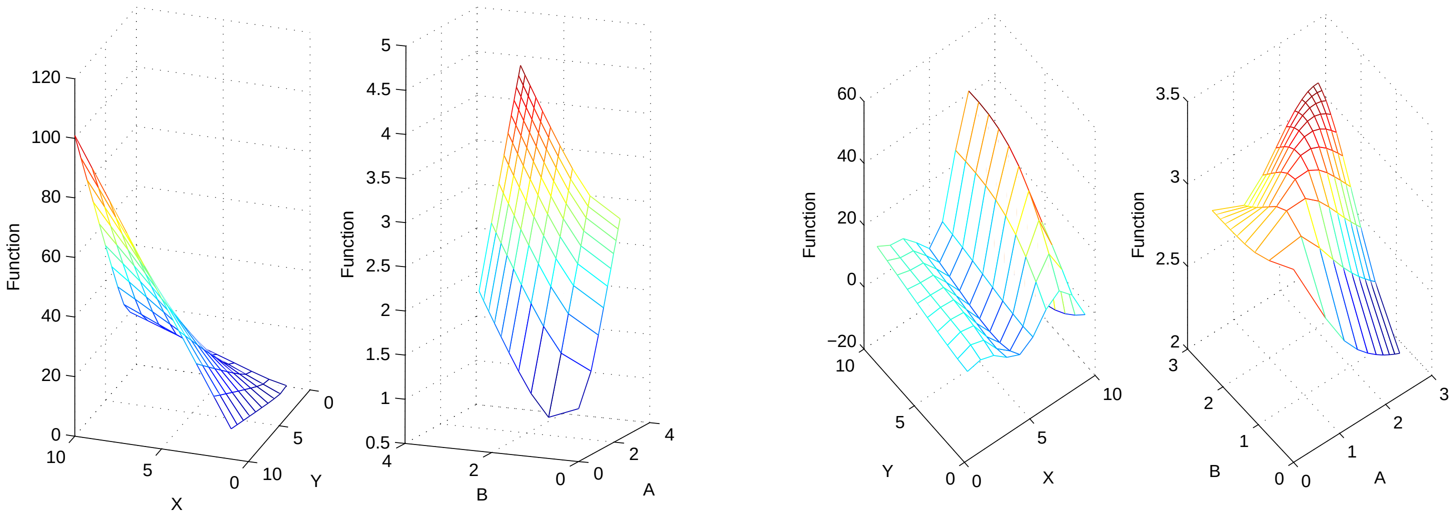
Any combination is GP in high SIR

Any combination without (C,D,c) is GP in any SIR

Objective Function	*Constraints*
(A) Maximize R_{i^*}	(a) $R_i \geq R_{i,min}$
(B) Maximize $\min_i R_i$	(b) $P_{i1}G_{i1} = P_{i2}G_{i2}$
(C) Maximize $\sum_i R_i$	(c) $\sum_i R_i \geq R_{system,min}$
(D) Maximize $\sum_i w_i R_i$	(d) $P_{o,i} \leq P_{o,i,max}$
(E) Minimize $\sum_i P_i$	(e) $0 \leq P_i \leq P_{i,max}$

GP and Convexity

Convexity is not invariant under nonlinear change of coordinates



GP and SOS

Minimize a sum of monomials with upper bound constraints on other sums

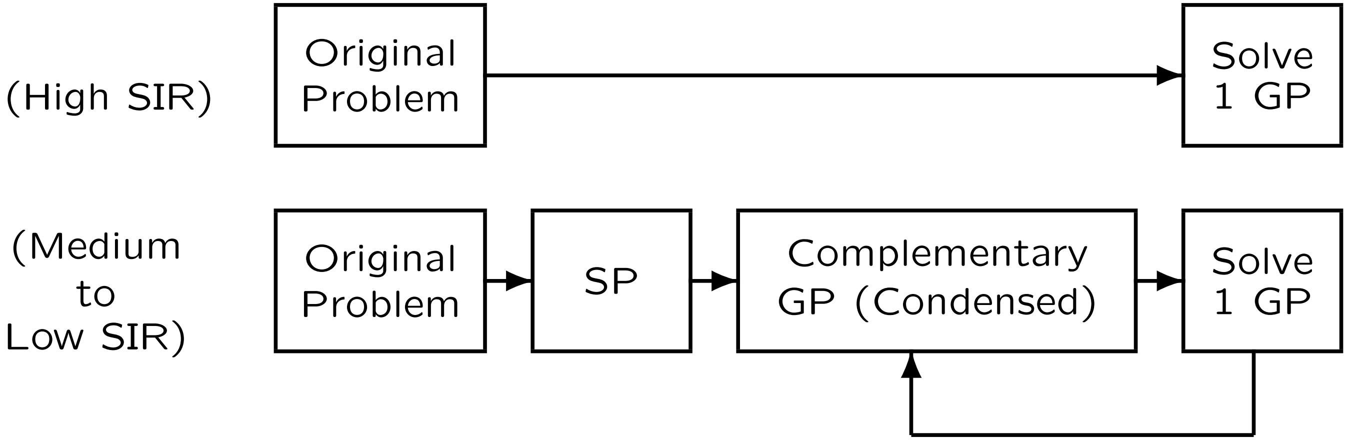
Different definitions of monomial: $c\prod_j x_j^{a^{(j)}}$

	GP	*PMoP*	*SP*
c	$\mathbf{R}_+$	$\mathbf{R}$	$\mathbf{R}$
$a^{(j)}$	$\mathbf{R}$	$\mathcal{Z}_+$	$\mathbf{R}$
x_j	$\mathbf{R}_{++}$	$\mathbf{R}_{++}$	$\mathbf{R}_{++}$

- GP (Polynomial time)
- PMoP: constrained polynomial minimization over the positive quadrant (SOS)
- SP: Signomial Programming (Condensation)

1 GP or Many GPs

SIR is an inverted posynomial in $\mathbf{P}$ but $1+\text{SIR}$ is not



SOS: successive SDP approximations

- Finite time convergence to global optimum

Condensation: successive GP approximations

- Asymptotic convergence to local optimum

Condensation Method

Lemma: $g(\mathbf{x}) = \sum_i u_i(\mathbf{x})$ is a posynomial

$$g(\mathbf{x}) \geq \tilde{g}(\mathbf{x}) = \prod_i \left(\frac{u_i(\mathbf{x})}{\alpha_i} \right)^{\alpha_i}$$

If $\alpha_i = u_i(\mathbf{x}_0)/g(\mathbf{x}_0), \ \forall i$, then $\tilde{g}(\mathbf{x}_0)$ is the best monomial approximation

Algorithm:

1) Evaluate the denominator posynomial of signomials with the given $\mathbf{P}$.

2) Compute for each term i in this posynomial,

$$\alpha_i = \frac{\text{value of } i\text{th term in posynomial}}{\text{value of posynomial}}.$$

3) Condense the denominator posynomial into a monomial.

4) Solve the resulting GP using an interior point method.

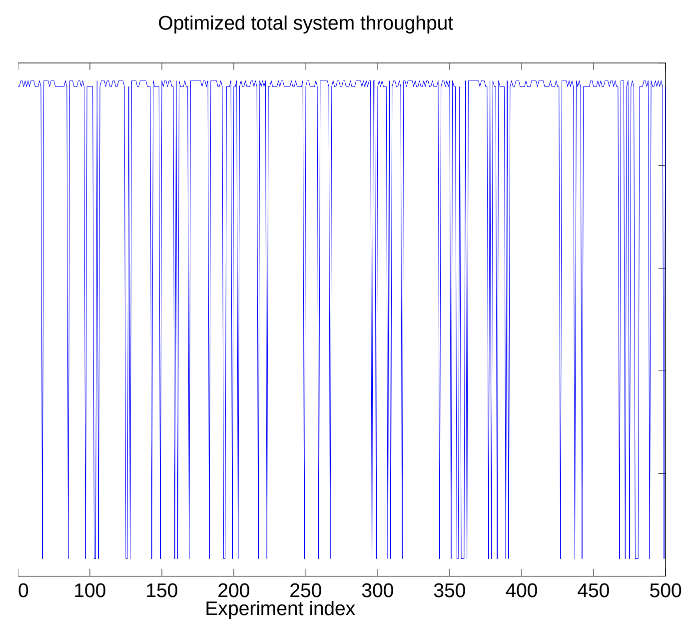
5) Go to step 1 using $\mathbf{P}$ of step 4.

6) Terminate the kth loop if $\| \mathbf{P}^{(k)} - \mathbf{P}^{(k-1)} \| \leq \epsilon$

Example

Problem formulation: System throughput maximization under individual user's rate and outage constraints

96% of the time: attains global optimum



Other Extensions

- Effective heuristics for jumping out of local optimum
- Signomial constraints in constraints

Distributed GP Power Control

Generally applicable to coupled but additive objective function:

$$\text{minimize} \quad \sum_i f_i(x_i, \{x_j\}_{j \in I(i)})$$

Log change of variables for convexity:

$$\text{minimize} \quad \sum_i f_i(e^{y_i}, \{e^{y_j}\}_{j \in I(i)})$$

Introduce auxiliary variables (local copies) and consistency constraints:

$$\begin{array}{ll} \text{minimize} & \sum_i f_i(e^{y_i}, \{e^{y_{ij}}\}_{j \in I(i)}) \\ \text{subject to} & y_{ij} = y_j, \forall j \in I(i), \forall i. \end{array}$$

Lagrangian relaxation for dual decomposition with consistency pricing:

$$\begin{aligned} L(\{y_i\}, \{y_{ij}\}; \{\gamma_{ij}\}) &= \sum_i f_i(e^{y_i}, \{e^{y_{ij}}\}_{j \in I(i)}) + \sum_i \sum_{j \in I(i)} \gamma_{ij}(y_j - y_{ij}) \\ &= \sum_i L_i(y_i, \{y_{ij}\}; \{\gamma_{ij}\}) \end{aligned}$$

Distributed GP Power Control

Lagrange dual problem:

$$\max_{\{\gamma_{ij}\}} \quad g(\{\gamma_{ij}\}) = \sum_i \min_{y_i,\{y_{ij}\}} L_i(y_i, \{y_{ij}\}; \{\gamma_{ij}\})$$

Dual ascent algorithm with consistency pricing update:

$$\gamma_{ij}(t+1) = \gamma_{ij}(t) + \delta(t)(y_j(t) - y_{ij}(t))$$

Message passing needed:

- $\mathcal{O}(N^2)$ as above
- $\mathcal{O}(N)$ by utilizing interference terms