# ELE539A: Optimization of Communication Systems
# Lecture 3A: Linear Programming

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February 12, 2007

## Lecture Outline

- Linear programming
- Norm minimization problems
- Dual linear programming
- Basic properties

Thanks: Stephen Boyd (some materials and graphs from Boyd and Vandenberghe)

## Linear Programming

Minimize linear function over linear inequality and equality constraints:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

Variables: $x \in \mathbf{R}^n$.

Standard form LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \succeq 0 \end{array}$$

Most well-known, widely-used and efficiently-solvable optimization

Appreciation-Application cycle starting for convex optimization

## Transformation To Standard Form

Introduce slack variables $s_i$ for inequality constraints:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx + s = h \\ & Ax = b \\ & s \succeq 0 \end{array}$$

Express $x$ as difference between two nonnegative variables $x^+, x^- \succeq 0$: $x = x^+ - x^-$

$$\begin{array}{ll} \text{minimize} & c^T x^+ - x^T x^- \\ \text{subject to} & Gx^+ - Gx^- + s = h \\ & Ax^+ - Ax^- = b \\ & x^+, x^-, s \succeq 0 \end{array}$$

Now in LP standard form with variables $x^+, x^-, s$

# Linear Fractional Programming

Minimize ratio of affine functions over polyhedron:

$$
\begin{array}{ll}
\text{minimize} & \frac{c^T x + d}{e^T x + f} \\
\text{subject to} & Gx \preceq h \\
& Ax = b
\end{array}
$$

Domain of objective function: $\{x | e^T x + f > 0\}$

Not an LP. But if nonempty feasible set, transformation into an equivalent LP with variables $y, z$:

$$
\begin{array}{ll}
\text{minimize} & c^T y + dz \\
\text{subject to} & Gy - hz \preceq 0 \\
& Ay - bz = 0 \\
& e^T y + fz = 1 \\
& z \succeq 0
\end{array}
$$

Why: let $y = \frac{x}{e^T x + f}$ and $z = \frac{1}{e^T x + f}$

## Norm Minimization Problems

- $l_1$ norm: $\|x\|_1 = \sum_{i=1}^n |x_i|$

Minimize $\|Ax - b\|_1$ is equivalent to this LP in $x \in \mathbf{R}^n, s \in \mathbf{R}^n$:

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T s \\ \text{subject to} & Ax - b \preceq s \\ & Ax - b \succeq -s \end{array}$$

- $l_\infty$ norm: $\|x\|_\infty = \max_i\{|x_i|\}$

Minimize $\|Ax - b\|_\infty$ is equivalent to this LP in $x \in \mathbf{R}^n, t \in \mathbf{R}$:

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & Ax - b \preceq t\mathbf{1} \\ & Ax - b \succeq -t\mathbf{1} \end{array}$$

# Dual Linear Programming

1. Primal problem in standard form:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \succeq 0 \end{array}$$

2. Write down Lagrangian using Lagrange multipliers $\lambda, \nu$:

$$L(x, \lambda, \nu) = c^T x - \sum_{i=1}^{n} \lambda_i x_i + \nu^T (Ax - b) = -b^T \nu + (c + A^T \nu - \lambda)^T x$$

3. Find Lagrange dual function:

$$g(\lambda, \nu) = \inf_x L(x, \lambda, \nu) = -b^T \nu + \inf_x [(c + A^T \nu - \lambda)^T x]$$

Since a linear function is bounded below only if it is identically zero, we have

$$g(\lambda, \nu) = \begin{cases} -b^T \nu & A^T \nu - \lambda + c = 0 \\ -\infty & \text{otherwise.} \end{cases}$$

# Dual Linear Programming

4. Write down Lagrange dual problem:

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) = \begin{cases} -b^T\nu & A^T\nu - \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases} \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

5. Make equality constraints explicit:

$$\begin{array}{ll} \text{maximize} & -b^T\nu \\ \text{subject to} & A^T\nu - \lambda + c = 0 \\ & \lambda \succeq 0 \end{array}$$

6. Simplify Lagrange dual problem:

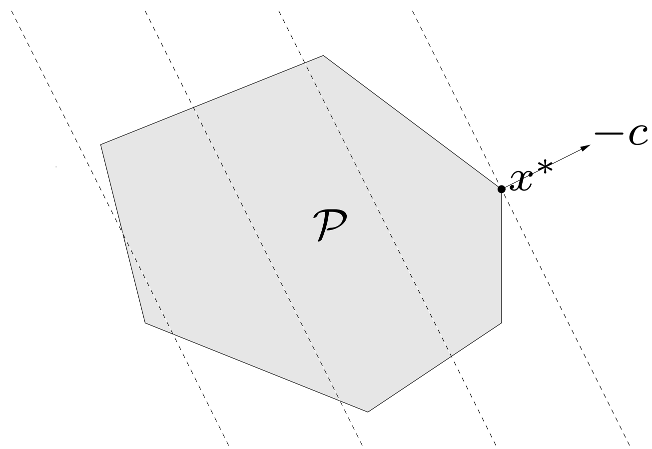
$$\begin{array}{ll} \text{maximize} & -b^T\nu \\ \text{subject to} & A^T\nu + c \succeq 0 \end{array}$$

which is an inequality constrained LP

## Basic Properties

Definition: $x$ in polyhedron $P$ is an extreme point if there does not exist two other points $y, z \in P$ such that $x = \theta y + (1-\theta)z$ for some $\theta \in [0,1]$

Theorem: Assume that a LP in standard form is feasible and the optimal objective value is finite. There exists an optimal solution which is an extreme point



## Algorithms

- Simplex Method
- Interior-point Method
- Ellipsoid Method
- Cutting-plane Method

Simplex method is very efficient in practice but specialized for LP: move from one vertex to another without enumerating all the vertices

We will cover interior point algorithms for general convex optimization later

## Lecture Summary

- LP covers a wide range of interesting problems for communication systems
- Dual LP is LP
- There are very useful special structures in LP. But most of the important ones (computational efficiency, global optimality, Lagrange duality) can be generalized to convex optimization
- After another lecture on network flow LP, we will study the applications of nonlinear convex optimization, then nonlinear nonconvex optimization

Readings: Section. 4.3, 5.1-5.2 of Boyd and Vanderberghe