ELE539A: Optimization of Communication Systems
Lecture 3B: Network Flow Problems

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Lecture Outline

- Network flow problems
- Problem 1: Maximum flow problem
- Ford Fulkerson algorithm
- Problem 2: Shortest path routing
- Bellman Ford algorithm
- Simple IP routing: RIP
- Dynamic Programming

Graph Theory Notation

$G = (V, E)$: directed graph with vertex set V and edge set E

b_i: external supply to each node $i \in V$

u_{ij}: capacity of each edge $(i, j) \in E$

c_{ij}: cost per unit flow on edge $(i, j) \in E$

$I(i) = \{j \in V | (j, i) \in E\}$: set of start nodes of incoming edges to i

$O(i) = \{j \in V | (i, j) \in E\}$: set of end nodes of outgoing edges from i

Sources: $\{i | b_i > 0\}$. Sinks: $\{i | b_i < 0\}$

Feasible flow f:

- Flow conservation: $b_i + \sum_{j \in I(i)} f_{ji} = \sum_{j \in O(i)} f_{ij}, \ \ \forall i \in V$
- Capacity constraint: $0 \leq f_{ij} \leq u_{ij}$

Basic Formulation

Network flow problem:

$$\begin{array}{ll} \text{minimize} & \sum_{(i,j)\in E} c_{ij} f_{ij} \\ \text{subject to} & b_i + \sum_{j\in I(i)} f_{ji} = \sum_{j\in O(i)} f_{ij}, \quad \forall i \in V \\ & 0 \le f_{ij} \le u_{ij} \end{array}$$

In matrix notation as a LP:

$$\begin{array}{ll} \text{minimize} & c^T f \\ \text{subject to} & Af = b \\ & 0 \preceq f \preceq u \end{array}$$

where $A \in \mathbf{R}^{|V|\times|E|}$ is defined as

$$A_{ik} = \begin{cases} 1, & i \text{ is the start node of edge } k \\ -1, & i \text{ is the end node of edge } k \\ 0, & \text{otherwise} \end{cases}$$

Special Cases

- Maximum flow problem (this lecture)
- Shortest path problem (this lecture)
- Transportation problem (uncapacitated bipartite graph)

$$
\begin{array}{ll}
\text{minimize} & \sum_{i,j} c_{ij} f_{ij} \\
\text{subject to} & \sum_{i=1}^{m} f_{ij} = d_j, \quad j = 1, \ldots, n \\
& \sum_{j=1}^{n} f_{ij} = s_i, \quad i = 1, \ldots, m \\
& f_{ij} \geq 0, \quad i = 1, \ldots, m, j = 1, \ldots, n
\end{array}
$$

Variables f_{ij}. Constants d_j, s_i, c_{ij}

- Assignment problem (homework):

$m = n, d_j = s_i = 1$ in transportation problem

Maximum Flow Problem

$$
\begin{array}{ll}
\text{maximize} & b_s \\
\text{subject to} & Af = b \\
& b_t = -b_s \\
& b_i = 0, \ \ \forall i \neq s, t \\
& 0 \leq f_{ij} \leq u_{ij}
\end{array}
$$

Reformulated as network flow problem:

- Costs for all edges are zero
- Introduce a new edge (t, s) with infinite capacity and cost -1
- Minimize total cost is equivalent to maximize f_{ts}

Ford Fulkerson Algorithm

1. Start with feasible flow f

2. Search for an augmenting path P

3. Terminate if no augmenting path

4. Otherwise, if flow can be pushed, push $\delta(P)$ units of flow along P and repeat Step 2

5. Otherwise, terminate

Q: How to find augmenting path?

Q: How much flow can be pushed?

Augmenting Path

Idea: find a path where we can increase flow along every forward edge and decrease flow along backward edge by the same amount. Still satisfy constraints. Increase objective function

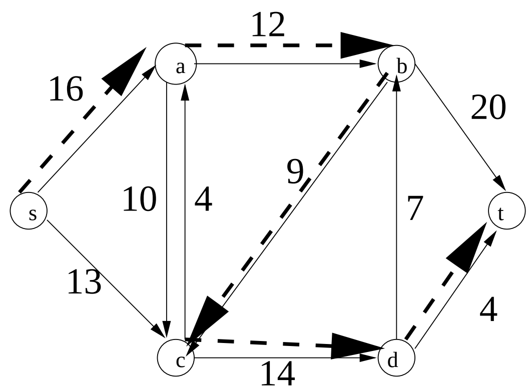
Augmenting path: a path from s to t such that $f_{ij} < u_{ij}$ on forward edges and $f_{ij} > 0$ on backward edges

Augmenting flow amount along augmenting path P:

$$\delta(P) = \min \left\{ \min_{(i,j) \in F} (u_{ij} - f_{ij}), \min_{(i,j) \in B} f_{ij} \right\}$$

Can search for augmenting path by following possible paths leading from s and checking conditions above

Example



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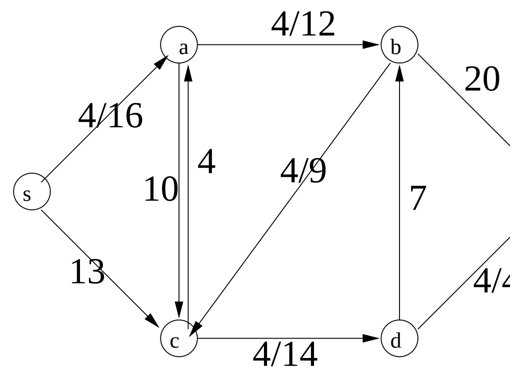

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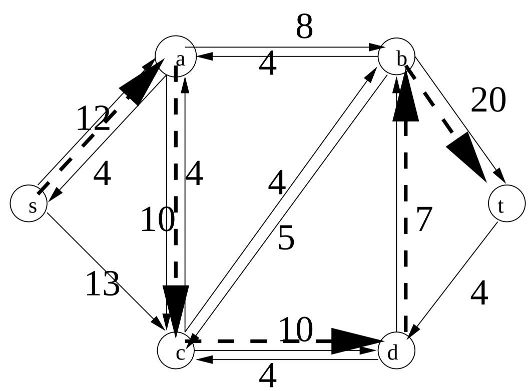

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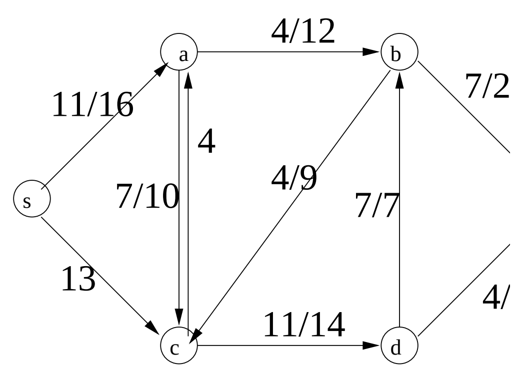

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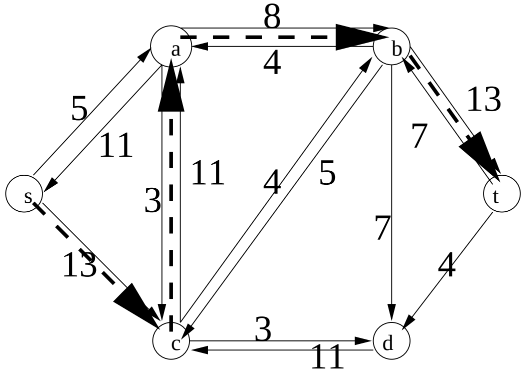
Example



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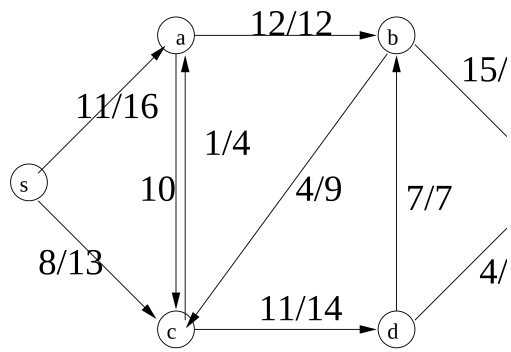

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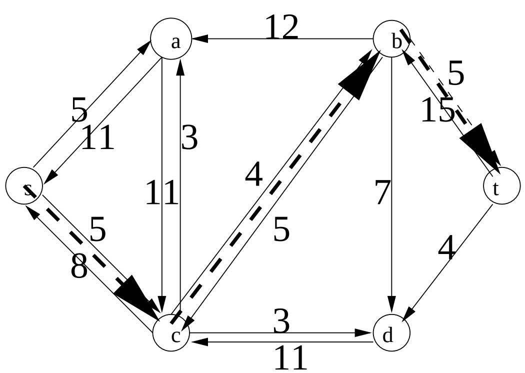

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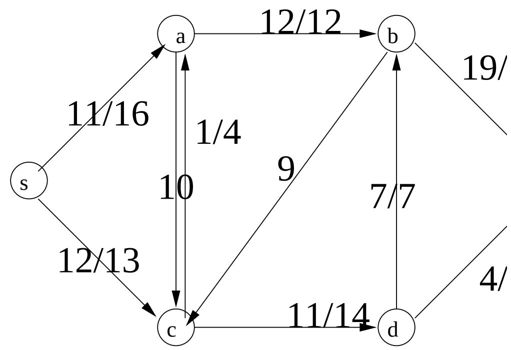

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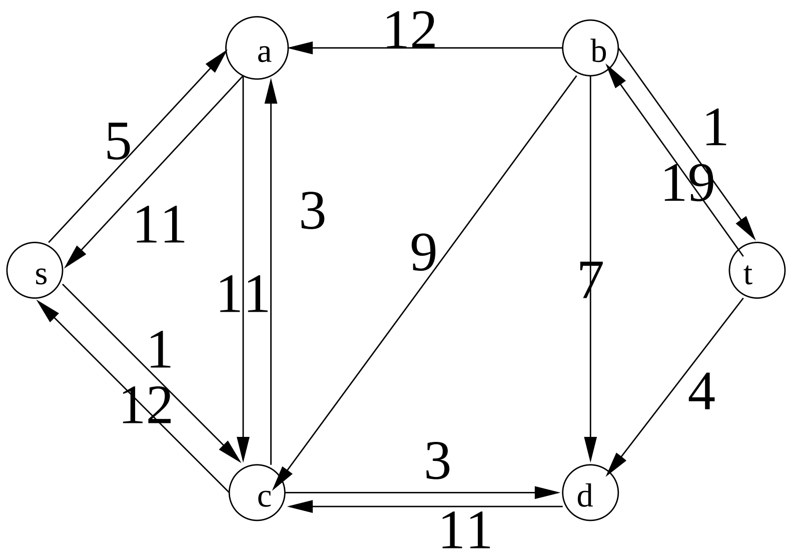
Example



Max Flow Min Cut Theorem

Theorem: If optimal value is finite, Ford Fulkerson algorithm terminates with an optimal flow

Theorem: If edge capacities u_{ij} are integers, edge flow variables remain integer

Definition: cut S is a subset of V such that $s \in S$ and $t \notin S$

Definition: capacity of cut $C(S)$ is sum of edge capacities on edges that cross from S to its complement:

$$C(S) = \sum_{(i,j) \in E | i \in S, j \notin S} u_{ij}$$

Theorem: Value of maximum flow $\max b_s$ equals minimum cut capacity $\min_S C(S)$

Shortest Path Routing

Given a directed graph with vertex set V and edge set E

Each edge (i, j) has cost or length c_{ij}

Allow negative length edges, but no negative length cycles

Our development follows DP algorithm

Other approaches (*e.g.*, duality) and algorithms (*e.g.*, Dijstrak) possible

Consider all-to-one shortest path routing with destination vertex n

Bellman Ford Algorithm

Let $p_i(t)$ be length of shortest path from i to n using at most t edges, with $p_i(t) = \infty$ if no such path exists

Let $p_n(t) = 0$, $\forall t$ and $p_i(0) = \infty$, $\forall i \neq n$

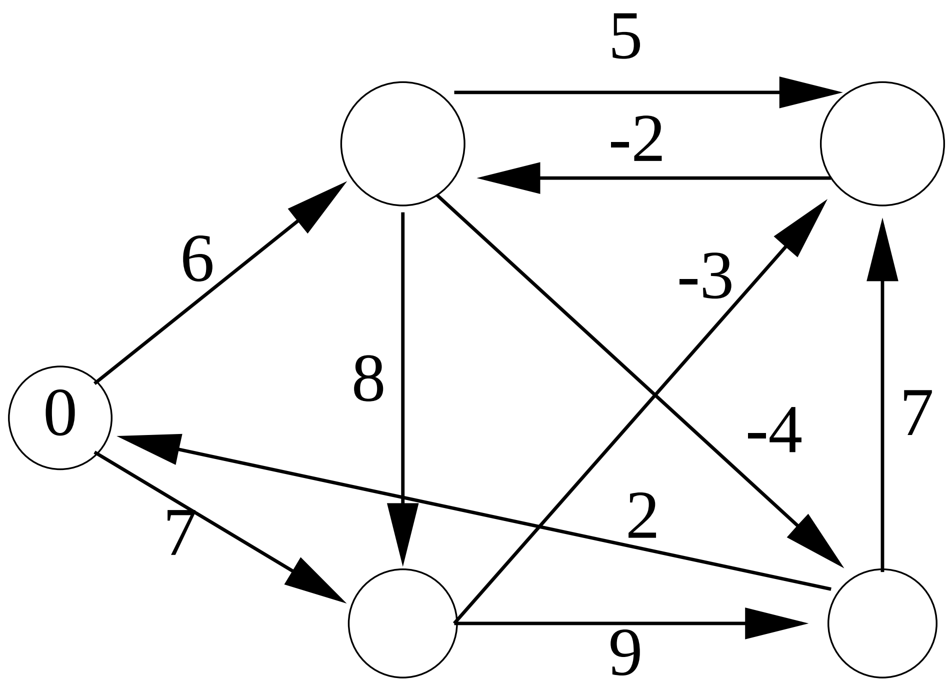
$p_i(t+1)$ consists of two parts:

- cost of getting from i to a neighboring k
- cost of getting from k to destination n

Pick the minimum total cost:

$$p_i(t+1) = \min_{k \in \mathcal{O}(i)} \{c_{ik} + p_k(t)\}$$

Example



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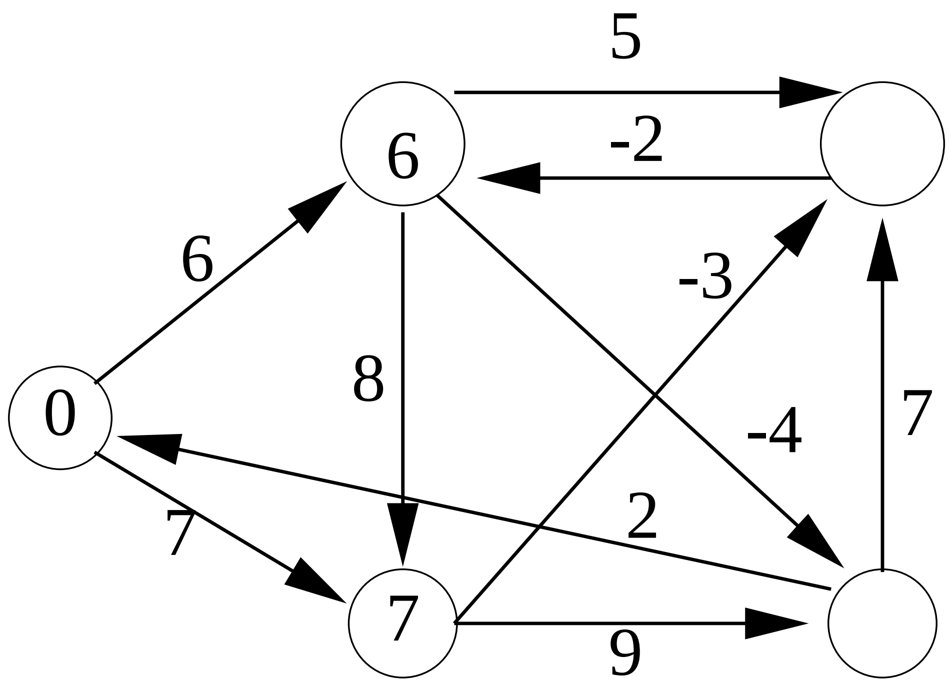

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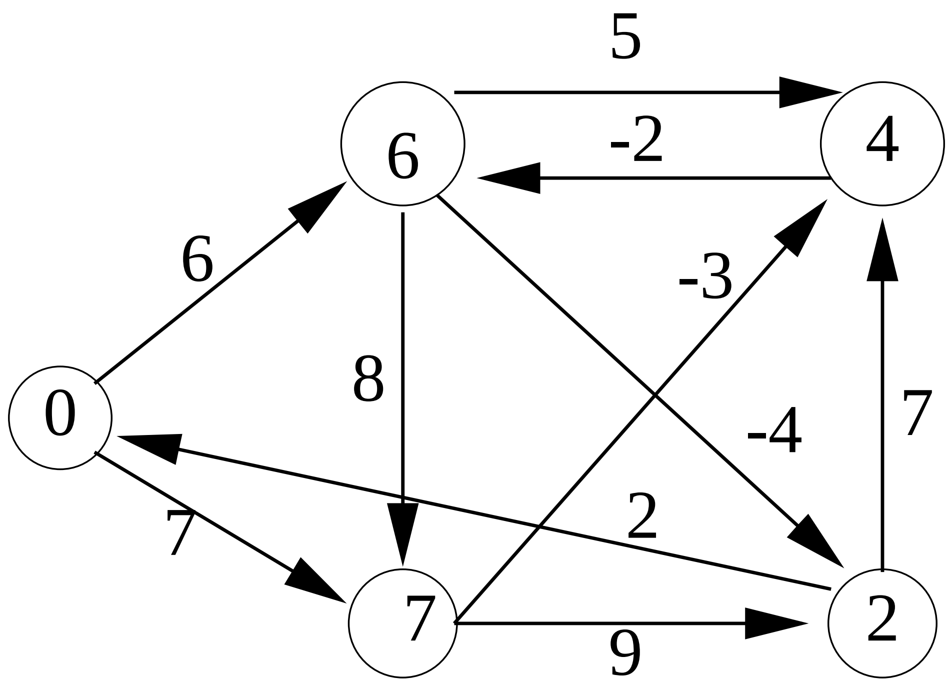

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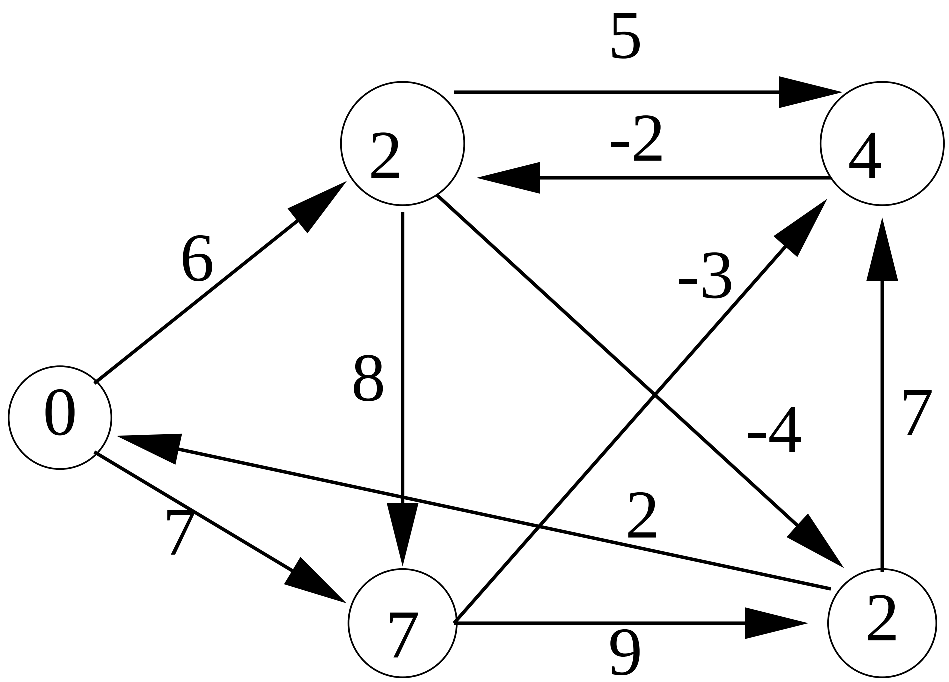
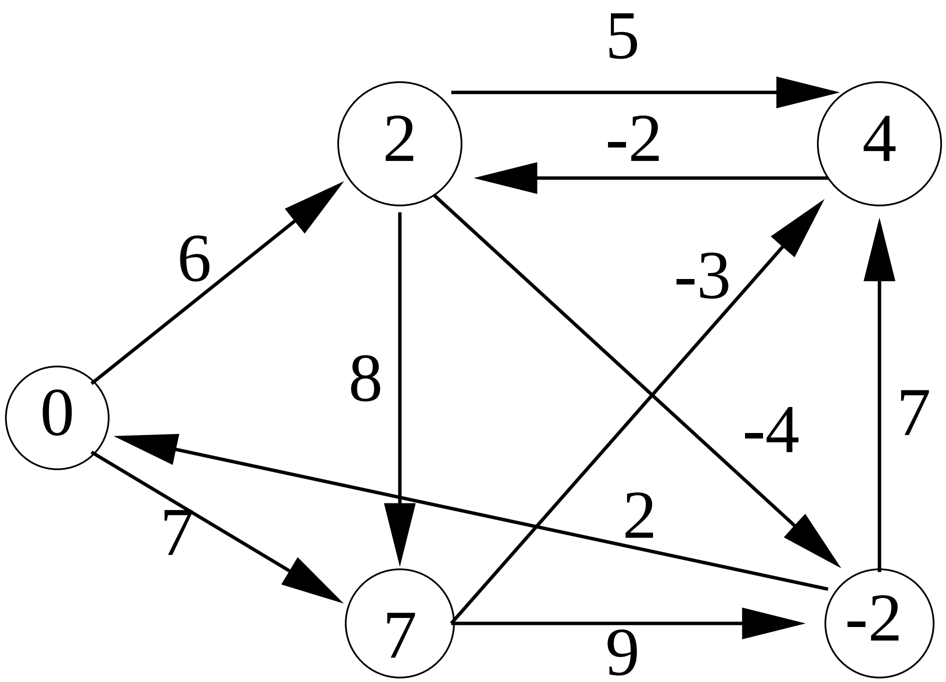


Example



IP Routing

Basic versions:

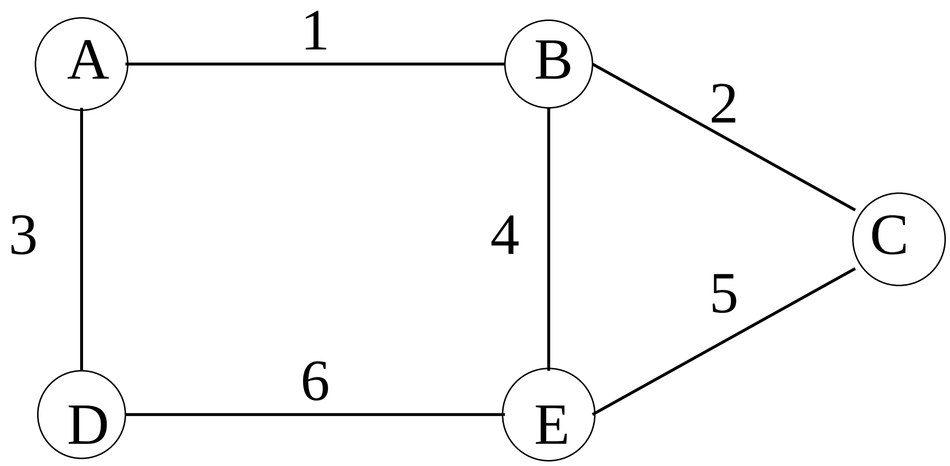
- IGP (*e.g.*, RIP): distance-vector based
- IGP (*e.g.*, OSPF, IS-IS): link-state based
- EGP (*e.g.*, BGP4): across Autonomous Systems

Extensions:

- Multicast routing
- Mobile IP
- Mobile wireless ad hoc routing
- QoS routing

RIP Routing

Simple example (homework):



Practical concerns:

- Loop avoidance
- Stability
- Speed of convergence
- Scalability

Sequential Optimization

Additive cost in discrete time dynamic system:

$$x_{k+1} = f_k(x_k, u_k, w_k), \;\; k = 0, \ldots, N-1$$

State: $x_k \in S_k$

Control: $u_k \in U_k(x_k)$

Random disturbance: $w_k \in D_k$ with distribution conditional on x_k, u_k

Admissible policies:

$$\pi = \{\mu_0, \ldots, \mu_{N-1}\}$$

where $\mu_k(x_k) = u_k$ such that $\mu_k(x_k) \in U_k(x_k)$ for all $x_k \in S_k$

Given cost functions $g_k, k = 0, \ldots, N$, expected cost of π starting at x_0:

$$J_\pi(x_0) = \mathbf{E}\left(g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, \mu_k(x_k), w_k)\right)$$

Optimal policy π^* minimizes J over all admissible π, with optimal cost:

$$J^*(x_0) = J_{\pi^*}(x_0) = \min_{\pi \in \Pi} J_\pi(x_0)$$

Principle of Optimality

Given optimal policy $\pi^* = \{\mu_0^*, \ldots, \mu_{N-1}^*\}$. Consider subproblem where at time i and state x_i, minimize cost-to-go function from time i to N:

$$\mathbf{E}\left(g_N(x_N) + \sum_{k=i}^{N-1} g_k(x_k, \mu_k(x_k), w_k)\right)$$

Then truncated optimal policy $\{\mu_i^*, \ldots, \mu_{N-1}^*\}$ is optimal for subproblem

Tail of an optimal policy is also optimal for tail of the problem

DP Algorithm

For every initial state x_0, $J^*(x_0)$ equals $J_0(x_0)$, the last step of the following backward iteration:

$$J_N(x_N) = g_N(x_N)$$

$$J_k(x_k) = \min_{u_k \in U_k(x_k)} \mathbf{E}\left(g_k(x_k, u_k, w_k) + J_{k+1}(f_k(x_k, u_k, w_k))\right), \quad k = 0, \ldots, N-1$$

If $\mu_k^*(x_k) = u_k^*$ are the minimizers of $J_k(x_k)$ for each x_k and k, then policy

$$\pi^* = \{\mu_0^*, \ldots, \mu_{N-1}^*\}$$

is optimal

Proof: induction and Principle of Optimality

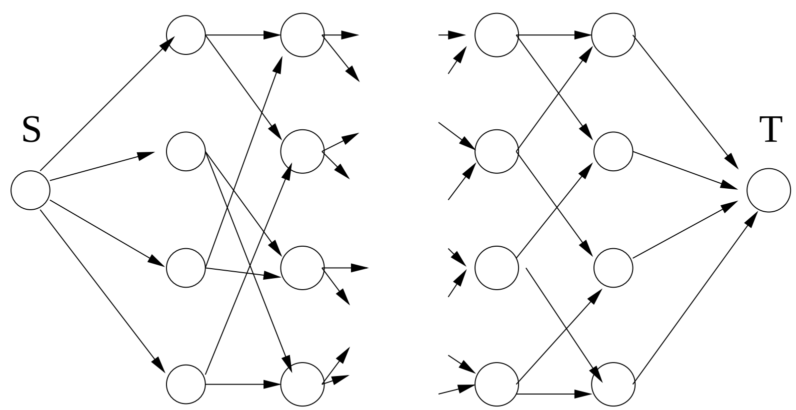
Deterministic Finite-State DP

- No stochastic perturbation:

$$x_{k+1} = f_k(x_k, \mu_k(x_k))$$

- Finite state space: S_k are finite for all k

Deterministic finite-state DP is equivalent to shortest path problem in trellis diagram



Lecture Summary

- Network flow problems are special cases of LP that model a wide range of problems in networking and problems modelled by graphs.
- Maximum flow problems and shortest path problems are two important special cases of network flow problems that can be efficiently solved by special purpose distributed algorithms.
- DP principle is extremely powerful for sequential optimization.
- We will later study powerful generalizations of Network, Flow Problems to Network Utility Maximization.
- Practical issues in IP routing (IGP and BGP) to be taught in Rexford guest lecture.

Reading: Section 7.1, 7.2, 7.5, and 7.9 in Bertsimas and Tsitsiklis