ELE539A: Optimization of Communication Systems
Lecture 9: Estimation and Detection Problems

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Lecture Outline

- ML estimation by norm minimization
- Covariance estimation by SDP
- Distribution estimation by convex optimization
- Chebyshev bound and signal detection by SDP
- Multiuser detection by SDP relaxation

Thanks: Stephen Boyd (Some materials from Boyd and Vandenberghe)

ML Estimation

Estimate $x \in \mathbf{R}^n$ from a set of measurements $y_i \in \mathbf{R}$:

$$y_i = a_i^T x + v_i, \quad i = 1, \ldots, m$$

where v_i are i.i.d. noise with distribution p on $\mathbf{R}$

Likelihood function is

$$\prod_{i=1}^{m} p(y_i - a_i^T x)$$

Maximum Likelihood estimate is any optimal point for the problem in x:

$$\text{maximize} \quad \sum_{i=1}^{m} \log p(y_i - a_i^T x)$$

Since many distributions are log-concave, the above problem are often convex optimization

Examples

- Gaussian noise: v_i are Gaussian with zero mean and variance σ^2:

$p(z) = (2\pi\sigma^2)^{-1/2} \exp(-z^2/2\sigma^2)$

Log-likelihood function: $-(1/2)\log(2\pi\sigma) - (1/2\sigma^2)\|Ax - y\|_2^2$

$$\hat{x} = \underset{x}{\operatorname{argmin}} \|Ax - y\|_2$$

Solution of least-squares

- Laplacian noise: v_i are Laplacian: $p(z) = (1/2a)\exp(\frac{-|z|}{a}),\ a > 0$

$$\hat{x} = \underset{x}{\operatorname{argmin}} \|Ax - y\|_1$$

Solution of $l - 1$ norm minimization

Covariance Estimation for Gaussian Distribution

$y \in \mathbf{R}^n$ is multivariate Gaussian with zero mean and covariance matrix $R = \mathbf{E}\, yy^T \in \mathbf{S}^n_{++}$ with distribution:

$$p_R(y) = (2\pi)^{-n/2} \det(R)^{-1/2} \exp(-y^T R^{-1} y/2)$$

Goal: estimate covariance R based on N i.i.d. samples $y_1, \ldots, y_N \in \mathbf{R}^n$

Sample covariance matrix

$$Y = \frac{1}{N} \sum_{k=1}^{N} y_k y_k^T$$

Log likelihood function $\log p_R(y_1, \ldots, y_N)$

$$l(R) = -(Nn/2)\log(2\pi) - (N/2)\log\det R - (N/2)\mathbf{tr}(R^{-1}Y)$$

Information matrix $S = R^{-1}$, log likelihood function is concave in S:

$$l(S) = -(Nn/2)\log(2\pi) + (N/2)\log\det S - (N/2)\mathbf{tr}(SY)$$

Covariance Estimation for Gaussian Distribution

ML estimate of S found by solving convex optimization:

$$\begin{array}{ll} \text{maximize} & \log\det S - \textbf{tr}(SY) \\ \text{subject to} & S \in \mathcal{S} \end{array}$$

where $\mathcal{S}$ is a convex constraint set of S based on prior information about R

e.g., bounds on R: $L \preceq R \preceq U$ ($U^{-1} \preceq S \preceq L^{-1}$)

e.g., condition number constraint on R: $\lambda_{max}(R) \leq \kappa_{max}\lambda_{min}(R)$ ($uI \preceq S \preceq \kappa_{max}uI$, variables: (S, u))

Distribution Estimation

Remove assumptions on parametrization of distribution

Random variable X with values in finite set $\{\alpha_1, \ldots, \alpha_n\}$

Distribution characterized by $p \in \mathbf{R}^n$ that satisfies $p \succeq 0, \mathbf{1}^T p = 1$

Prior information about p from linear constraints, *e.g.*

$$\mathbf{E}\, X = \sum_{i=1}^{n} \alpha_i p_i = \alpha, \quad \mathbf{E}\, X^2 = \sum_{i=1}^{n} \alpha^2 p_i = \beta, \quad \sum_{i:\alpha_i \geq 0} p_i \leq 0.3$$

and from nonlinear constraints, *e.g.*

$$\mathbf{var}(X) = \sum_{i=1}^{n} \alpha_i^2 p_i - \left(\sum_{i=1}^{n} \alpha_i p_i \right)^2$$

which can be lower bounded and form a convex constraint

Objectives

Given convex constraints $\mathcal{P}$ on p, can optimize for different criteria

- Maximum likelihood: Observe N independent samples, with k_i samples with value α_i

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^{n} k_i \log p_i \\ \text{subject to} & p \in \mathcal{P} \end{array}$$

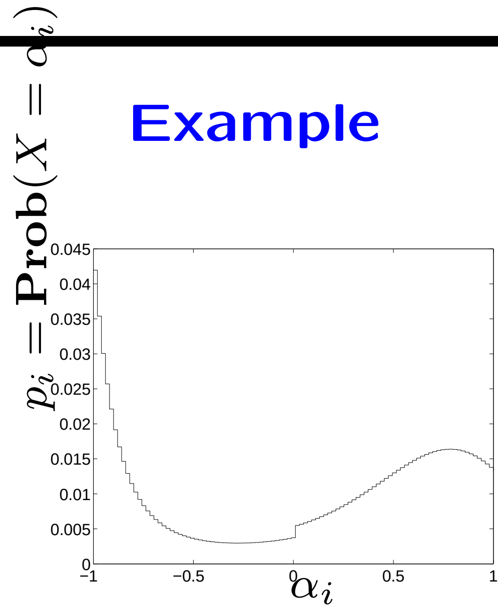
- Maximum entropy:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} p_i \log p_i \\ \text{subject to} & p \in \mathcal{P} \end{array}$$

- Minimum Kullback Leibler divergence with respect to another distribution q:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n} p_i \log(p_i / q_i) \\ \text{subject to} & p \in \mathcal{P} \end{array}$$

Example



Consider distribution on 100 equidistant points α_i in $[-1, 1]$ with the following prior information on p:

$$
\begin{aligned}
\mathbf{E}\, X &\in [-0.1, 0.1] \\
\mathbf{E}\, X^2 &\in [0.5, 0.6] \\
\mathbf{E}(3X^2 - 2X) &\in [-0.3, -0.2] \\
\mathbf{Prob}(X < 0) &\in [0.3, 0.4]
\end{aligned}
$$

Maximal entropy distribution satisfying these prior information can be computed by convex optimization

Chebyshev Bound with Second Order Prior Information (Continue LP Lecture Note)

Given prior information $\mathbf{E}\,X = a \in \mathbf{R}^m$ and $\mathbf{E}\,XX^T = \Sigma \in \mathbf{S}^m$

Express f as a general quadratic function:

$$f(z) = z^T P z + 2q^T z + r$$

with variables $P \in \mathbf{S}^m$, $q \in \mathbf{R}^m$, $r \in \mathbf{R}$

Objective to be minimized:

$$\mathbf{E}\,f(X) = \mathbf{tr}(\Sigma P) + 2q^T a + r$$

Constraint $f(z) \geq 0$ expressed as LMI:

$$\begin{bmatrix} P & q \\ q^T & r \end{bmatrix} \succeq 0$$

Chebyshev Bound by SDP

Set C we are interested is complement of polyhedron $\mathcal{P}$:

$$C = \mathbf{R}^m - \mathcal{P}, \quad \mathcal{P} = \{z | a_i^T z < b_i, i = 1, \ldots, k\}$$

Condition $f(z) \geq 1$ for all $z \in C$ is equivalent to

$$a_i^T z \geq b_i \Rightarrow z^T P z + 2q^T z + r \geq 1, \;\; i = 1, \ldots, k$$

which is equivalent to: there exist $\tau_1, \ldots, \tau_k \geq 0$ such that

$$\left[\begin{array}{cc} P & q \\ q^T & r - 1 \end{array} \right] \succeq \tau_i \left[\begin{array}{cc} 0 & a_i/2 \\ a_i^T/2 & -b_i \end{array} \right], \;\; i = 1, \ldots, k$$

Chebyshev bound problem as SDP in P, q, r, τ:

$$\begin{array}{ll} \text{minimize} & \mathbf{tr}(\Sigma P) + 2q^T a + r \\ \text{subject to} & \begin{bmatrix} P & q \\ q^T & r-1 \end{bmatrix} \succeq \tau_i \begin{bmatrix} 0 & a_i/2 \\ a_i^T/2 & -b_i \end{bmatrix}, \quad i = 1, \ldots, k \\ & \begin{bmatrix} P & q \\ q^T & r \end{bmatrix} \succeq 0 \\ & \tau \succeq 0 \end{array}$$

Optimal value is an upper bound on $\mathbf{Prob}(X \in C)$

Signal Detection Example

Set of m possible signals $s \in \{s_1, \ldots, s_m\}$. One of them transmitted through a noisy channel:

$$x = s + v$$

where v is noise with zero mean and covariance matrix $\sigma^2 I$

Minimum distance detector chooses s_k closest to x in Euclidean distance (ML decoding if v is Gaussian)

Correct detection condition for detecting s_k :

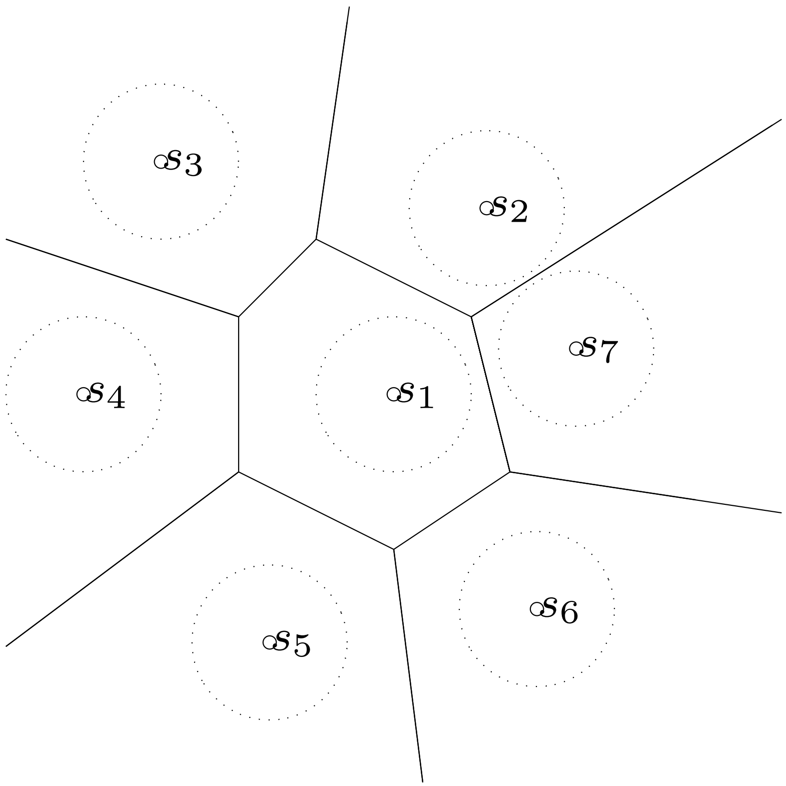
$$\|x - s_k\|_2 < \|x - s_j\|_2, \;\; j \neq k$$

Voronoi region V_k of s_k:

$$2(s_j - s_k)^T (s_k + v) < \|s_j\|_2^2 - \|s_k\|_2^2, \;\; j \neq k$$

Signal Detection Example

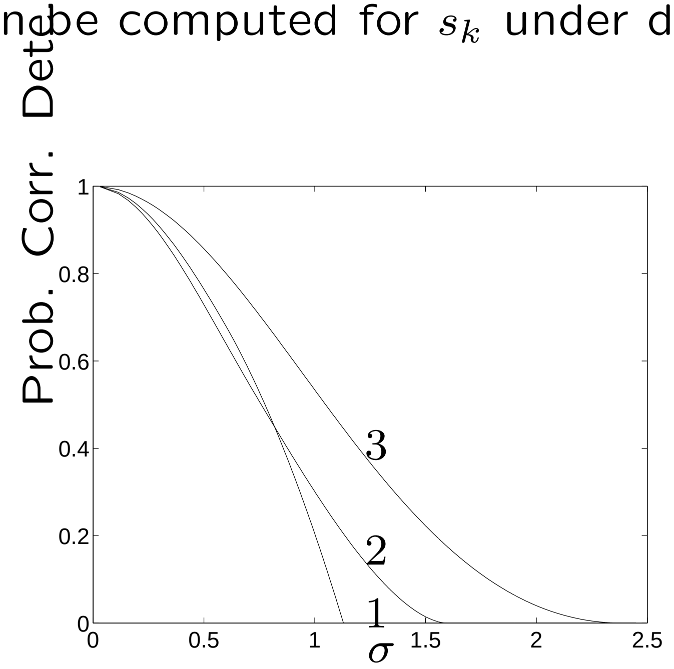
Signal constellation $s_i \in \mathbf{R}^2, i = 1, \ldots, 7$, with Voronoi regions shown



Signal Detection Example

Probability of correct detection: $\mathbf{Prob}(s_k + v \in V_k)$ (a polyhedron)

Chebyshev bound can be computed for s_k under different σ^2 by SDP:



Upper bound P_e for all noise with 0 mean and $\sigma^2 I$ covariance

Multi-user Detection

Received signal of K-user basic synchronous CDMA channel:

$$y(t) = \sum_{k=1}^{K} A_k b_k s_k(t) + n(t), \quad t \in [0, T]$$

Amplitude A_k, signature waveform $s_k(t)$, information bit $b_k \in \{-1, +1\}$, noise $n(t)$, period T

ML detection: find b that minimizes

$$\begin{aligned} & \int_0^T \left[y(t) - \sum_{k=1}^{K} A_k b_k s_k(t) \right]^2 dt \\ = \quad & \int_0^T \left[\sum_{k=1}^{K} A_k b_k s_k(t) \right] dt - 2 \int_0^T \left[\sum_{k=1}^{K} A_k b_k s_k(t) \right] y(t) dt \\ = \quad & b^T H b - 2 b^T A y \end{aligned}$$

Boolean Constrained QP Formulation

$A = \mathbf{diag}(A)$: amplitude matrix

$R_{ij} = \int_0^T s_i(t)s_j(t)dt$: cross-correlation matrix

$y = RAb + n$: sampled matched filter output

Notation: $H = ARA$ and $p = -2Ay$

$$\begin{array}{ll} \text{minimize} & x^T H x + p^T x \\ \text{subject to} & x_i \in \{-1, +1\}, \;\; i = 1, \ldots, K \end{array}$$

Equivalent form: let $X = xx^T$

$$\begin{array}{ll} \text{subject to} & X_{ii} = 1, \;\; i = 1, \ldots, K, \\ & X \succeq 0, \;\; \text{rank}(X) = 1 \end{array}$$

SDP Relaxation

Neglect rank constraint:

$$\begin{array}{ll} \text{subject to} & X_{ii} = 1, \;\; i = 1, \ldots, K, \\ & X \succeq 0 \end{array}$$

Let $\hat{X} = \begin{bmatrix} xx^T & x \\ x^T & 1 \end{bmatrix}$ and $C = \begin{bmatrix} H & p/2 \\ p^T/2 & 1 \end{bmatrix}$, reformulate as SDP:

$$\begin{array}{ll} \text{minimize} & \text{tr}(C\hat{X}) \\ \text{subject to} & \hat{X}_{ii} = 1, \;\; i = 1, \ldots, K+1, \\ & \hat{X} \succeq 0 \end{array}$$

Factorization and randomization methods to recover solution to original boolean constrained QP from SDP relaxation solutions

Dual Relaxation

Relax $x^T x = K$ to $x^T x \leq K$:

$$\begin{array}{ll} \text{minimize} & x^T H x + p^T x \\ \text{subject to} & x^T x \leq K \end{array}$$

Lagrange dual problem:

$$\begin{array}{ll} \text{minimize} & -\frac{1}{4} p^T (H + \lambda I)^{-1} p - \lambda K \\ \text{subject to} & \lambda \geq 0 \end{array}$$

Gradient-descent method solves this convex optimization in one variable λ

Recover primal optimal solution: $x^* = (H + \lambda^* I)^{-1} A y$

Complexity Performance Tradeoff

Exhaustive search: NP-hard $\mathcal{O}(2^K)$

Successive interference cancellation

Relaxation methods:

- SDP relaxation: polynomial time $\mathcal{O}(K^3)$ (SDP solution dominates randomization in terms of computational load)
- Unconstrained relaxation: analytic solution (related to decorrelator and linear MMSE)
- Bound relaxation
- Dual relaxation

Better complexity-tradeoff possible through duality, randomization, and other relaxation methods?

Lecture Outline

- Estimating probability distributions and detecting signals through various forms of convex optimization
- Extend Chebyshev bounds
- Relaxation for multi-user detection problems

Readings: Ch. 7.1-7.4 in Boyd and Vandenberghe.

W. K. Ma, T. N. Davidson, K. M. Wong, Z. Q. Luo, and P. C. Ching, "Quasi-maximum-likelihood multiuser detection using semidefinite relaxation with applications to synchronous CDMA," *IEEE Trans. Signal Proc.*, vol. 50, no. 4, pp. 912-922, April 2002.