An Axiomatic Theory of Fairness

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Acknowledgements

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- Evolving full version: www.princeton.edu/~chiangm/fairness.pdf
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Quantifying Fairness?

- x = [1 2 3]
  - How fair: 0.33, 0.86, ...?
- y = [1 10 100]
  - How fair: 0.01, 0.41, ...?
- y "looks like" less fair
  - How much less fair is y compared to x?

How Fair ?

- Given a vector x, how fair is it?
- Let $f : \mathcal{R}_+^n \to \mathcal{R}$ be a fairness measure
  - If f(x)>f(y), x is more fair than y
  - Full ordering and scale
- What axioms should it satisfy?

Approach 1: Index

- (often normalized) Measure of variance
  - Min-max ratio
  - Standard deviation
  - Entropy function $-\sum_i \frac{x_i}{\sum_j x_j} \log \frac{x_i}{\sum_j x_j}$
  - Jain's index $\frac{(\sum_i x_i)^2}{n \sum_i x_i^2}$
  - Diversity indices (e.g., Atkinson)

Approach 2: Objective Function

- alpha-fairness & alpha-fair utility function

$$U_\alpha(x) = \begin{cases} \frac{x^{1-\alpha}}{1-\alpha} & \alpha \geq 0,\ \alpha \neq 1 \\ \log(x) & \alpha = 1 \end{cases}$$

$$\sum_i \frac{x_i - y_i}{x_i^\alpha} \leq 0, \ \ \forall y$$

Special case: proportional and max-min fair

Questions

- Index and objective function approaches are different. Unify them?
- How many are there? Can I invent a new, MyLastName-Fairness?
- What cannot be a measure of fairness?
- How to enforce consistency despite diversity?
- How can I control "resolution" of fairness?
- Why is larger alpha more fair?

Axiomatization

ax·i·om

Pronunciation: \ˈak-sē-əm\

Function: *noun*
Date: 15th century
1 : a maxim widely accepted on its intrinsic merit
2 : a statement accepted as true as the basis for argument or inference
3 : an established rule or principle or a self-evident truth

ax·i·omat·i·za·tion

Pronunciation: \ˌak-sē-ə-ˌma-tə-ˈzā-shən, -sē-ˌä-mə-tə-\

Function: *noun*
Date: 1931
the act or process of reducing to a system of axioms

1. Axiom of Continuity

- $f(x)$ is continuous

- Small perturbation in resource allocation changes fairness measure's value slightly

2. Axiom of Homogeneity

- f(x) is a homogenous function of degree 0:

$$f(t\mathbf{x}) = f(\mathbf{x}), \quad \forall t > 0$$

- Unit/Magnitude doesn't matter
- More delicate than it might seem to be

3. Axiom of Saturation

- Equal allocation's fairness value is independent of number of users as the number of users becomes large:

$$\lim_{n\to\infty} \frac{f(\mathbf{1}_{n+1})}{f(\mathbf{1}_n)} = 1$$

- A "technical" axiom for uniqueness

4. Axiom of Partition

$$\mathbf{x} = [\mathbf{x}^1, \mathbf{x}^2], \quad \mathbf{y} = [\mathbf{y}^1, \mathbf{y}^2], \quad \sum_j x_j^i = \sum_j y_j^i$$

$$\frac{f(\mathbf{x})}{f(\mathbf{y})} = \mathrm{mean}\left(\frac{f(\mathbf{x}^1)}{f(\mathbf{y}^1)}, \frac{f(\mathbf{x}^2)}{f(\mathbf{y}^2)}\right)$$ for all partitions

- Well-definedness (of fairness value scale) growing from 2-user to n-user

Generator and Mean

- Mean function

$$\text{mean} = g^{-1}\left(\sum_i s_i g\left(\frac{f(\mathbf{x}^i)}{f(\mathbf{y}^i)}\right)\right)$$

  - "On mean values", Aczel (1948)

- g Kolmogorov-Nagumo generator
- $s_i = \left(\sum_{k\in i} x_k\right)^\rho$ Weight (normalized)

5. Axiom of Starvation

- $$\frac{f(0,1)}{f(\frac{1}{2},\frac{1}{2})} \leq 1$$
- 2-user: starvation is not more fair than equal distribution

5 Axioms

- Continuity
- Homogeneity (value statement)
- Saturation
- Partition
- Starvation (value statement)

So What?

- All axioms are "True"
- All axioms are "incorrect"
- Some are more "useful" than others

Useful Axiomatic System

- Non-redundant
- Self-consistent
- Unify known notions
- Discover new notions
- Provide useful insights

Existence & Uniqueness

- There exists a fairness measure satisfying Axioms 1-5
  - A "possibility theorem" on generator
- There is only one fairness function satisfying Axioms 1-5
  - Only log and power functions are possible generators $\{\log y, \quad y^\beta\}$

Existence Proof Outline

- By additive number-theoretic function property, show f(1) is independent of g
- Parameterize f for 2-user case
- Inductively go to N-user case
- Plug in some g to verify all axioms, especially Axioms 4 and 5
- (Local) uniqueness along the way

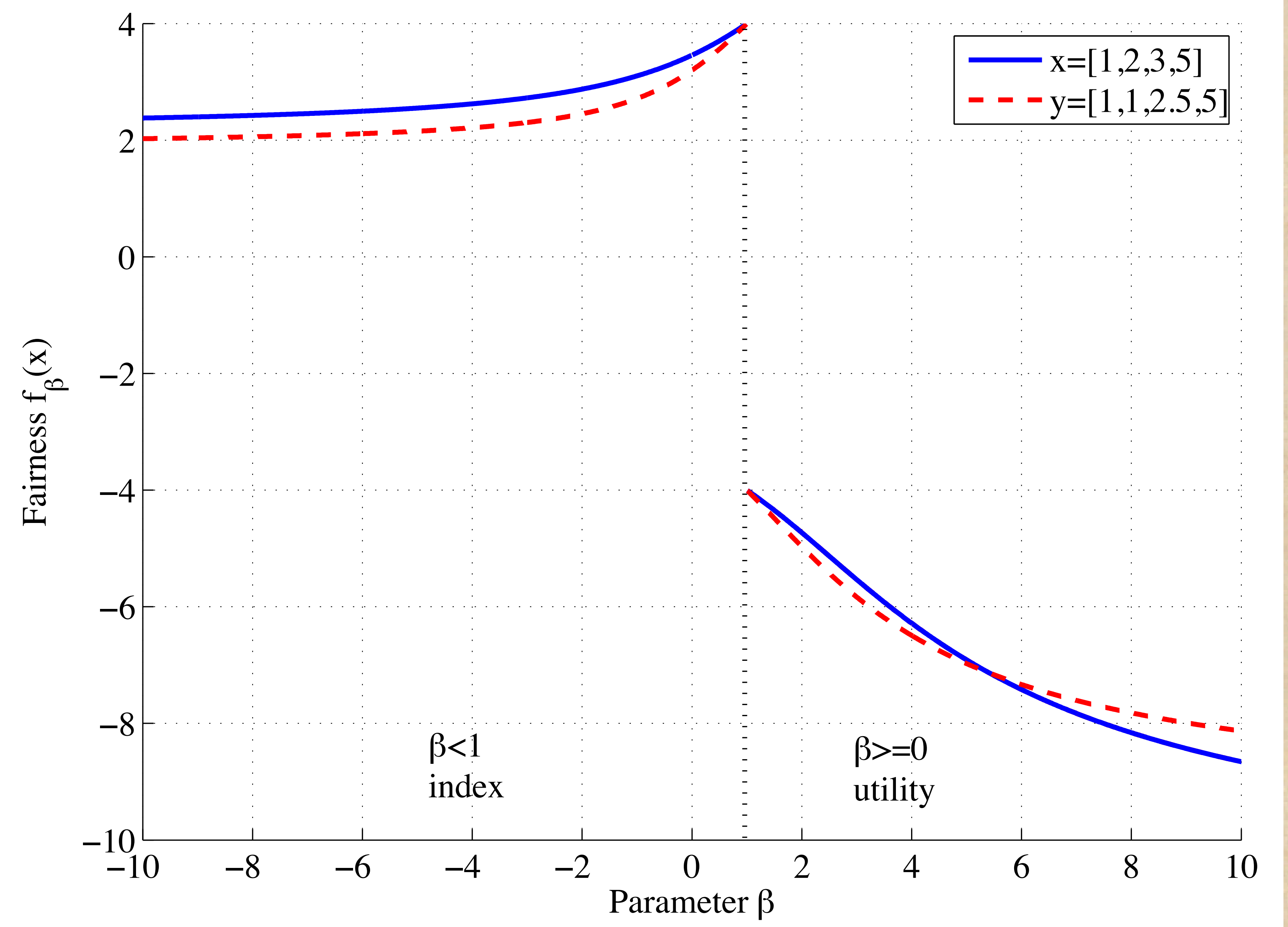
Constructed f

$$f_{\beta,r}(\mathbf{x}) = \operatorname{sign}(r(1-\beta r)) \left[\sum_{i=1}^{n} \left(\frac{x_i}{\sum_j x_j}\right)^{1-\beta r}\right]^{\frac{1}{\beta}}$$

$$r = \frac{1-\rho}{\beta} \qquad f(\mathbf{1}_n) = n^r \cdot f(1)$$

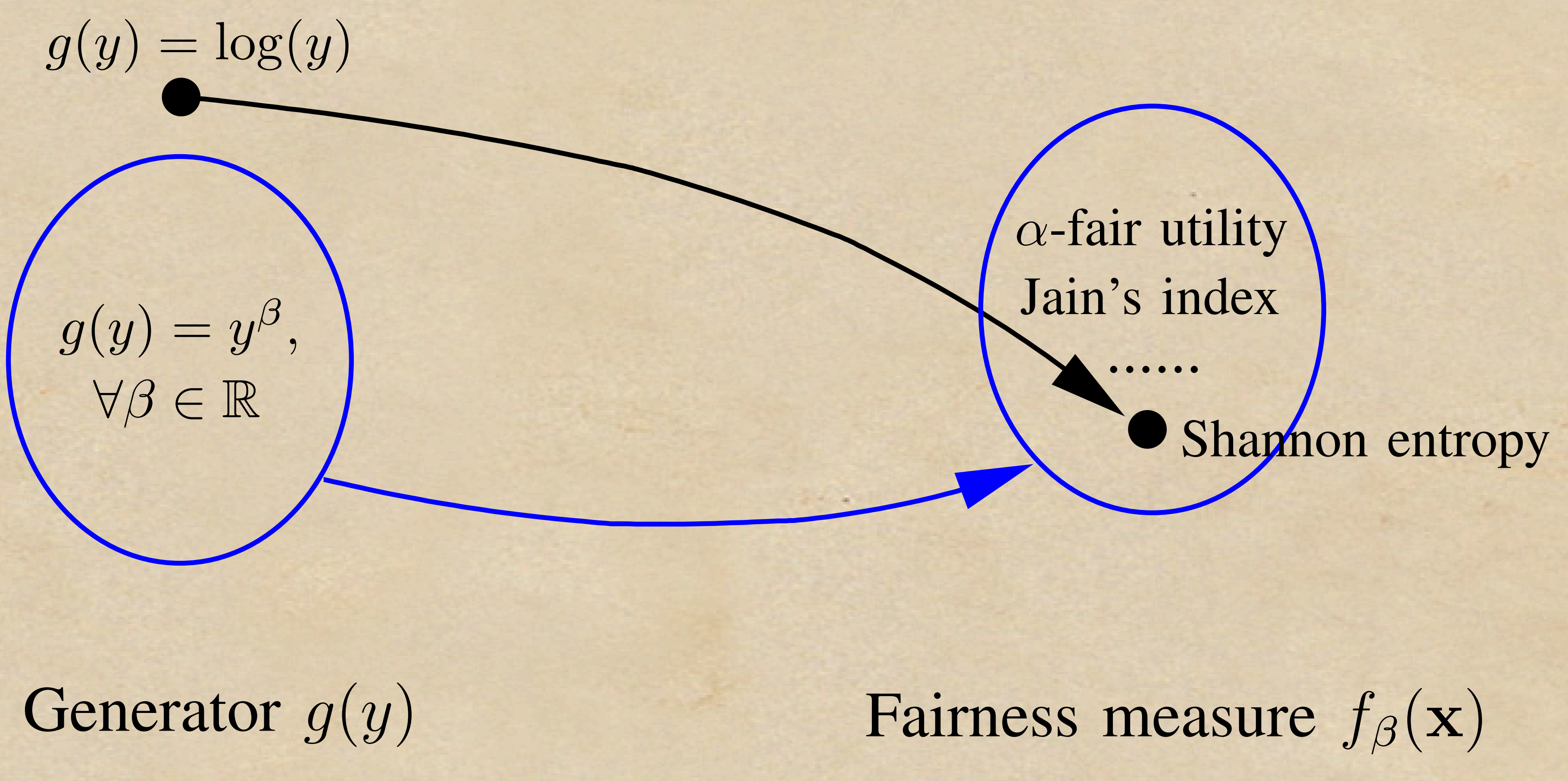
$$r = 1 \qquad f_{\beta}(\mathbf{x}) = \operatorname{sign}(1-\beta) \cdot \left[\sum_{i=1}^{n} \left(\frac{x_i}{\sum_j x_j}\right)^{1-\beta}\right]^{\frac{1}{\beta}}$$

An Example



x=[1,2,3,5]
y=[1,1,2.5,5]
β<1
index
β>=0
utility
Fairness $f_\beta(x)$
Parameter β
4
2
0
−2
−4
−6
−8
−10
−10
−8
−6
−4
−2
0
2
4
6
8
10

Generator -> Fairness



Unification (part 1)

Value of β	Our Fairness Measure	Known Names
$\beta \to \infty$	$-\max_i \left\{ \frac{\sum_i x_i}{x_i} \right\}$	Max ratio
$\beta \in (1, \infty)$	$-\left[(1-\beta) U_{\alpha=\beta} \left(\frac{\mathbf{x}}{w(\mathbf{x})} \right) \right]^{\frac{1}{\beta}}$	α-fair utility
$\beta = 1$	$\pm n$ (discontinuous)	No name
$\beta \in (0, 1)$	$\left[(1-\beta) U_{\alpha=\beta} \left(\frac{\mathbf{x}}{w(\mathbf{x})} \right) \right]^{\frac{1}{\beta}}$	α-fair utility
$\beta \to 0$	$e^{H\left(\frac{\mathbf{x}}{w(\mathbf{x})} \right)}$	Entropy
$\beta \in (0, -1)$	$\left[\sum_{i=1}^{n} \left(\frac{x_i}{w(\mathbf{x})} \right)^{1-\beta r} \right]^{\frac{1}{\beta}}$	No name
$\beta = -1$	$\frac{(\sum_i x_i)^2}{\sum_i {x_i}^2} = n \cdot J(\mathbf{x})$	Jain's index
$\beta \in (-1, -\infty)$	$\left[\sum_{i=1}^{n} \left(\frac{x_i}{w(\mathbf{x})} \right)^{1-\beta r} \right]^{\frac{1}{\beta}}$	No name
$\beta \to -\infty$	$\min_i \left\{ \frac{\sum_i x_i}{x_i} \right\}$	Min ratio

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Main Properties

- Symmetry $f(x,y) = f(y,x)$
- Equal allocation fairness value independent of g
- Equal allocation is most fair
- Constant tax reduces fairness
  - $f(x-c,y-c) <= f(x,y)$

Majorization...

- Majorization (a partial order on $\mathcal{R}^n$)
    - y majorizes x $\mathbf{x} \preceq \mathbf{y}$ if

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

$$\sum_{i=1}^{d} x_i^{\uparrow} \leq \sum_{i=1}^{d} y_i^{\uparrow}, \text{ for } d = 1, \ldots, n,$$

- Schur-concavity
    - $h(\mathbf{x}) \leq h(\mathbf{y}), \text{ if } \mathbf{x} \preceq \mathbf{y}$

… and Fairness

- Fairness measures satisfying Axioms 1 - 5 are Schur-concave
- A new ordering of Lorenz curves
  - "Standard" order: Gini (2 x area between Lorenz curve & straightline)
  - Axioms of Gini: Aaberge 2001
  - If x and y are majorizable => f and Gini give same order

More Properties

- Fairness value bounds the number of active users
- Fairness value bounds the maximum resource to a user
- Box constraints of resources allocation bound fairness value
- Perturbation of fairness value from slight change in a user's resource

Implications

- Make use of “new” fairness measures
- Understand properties of “old” ones

Generalized Jain's Index

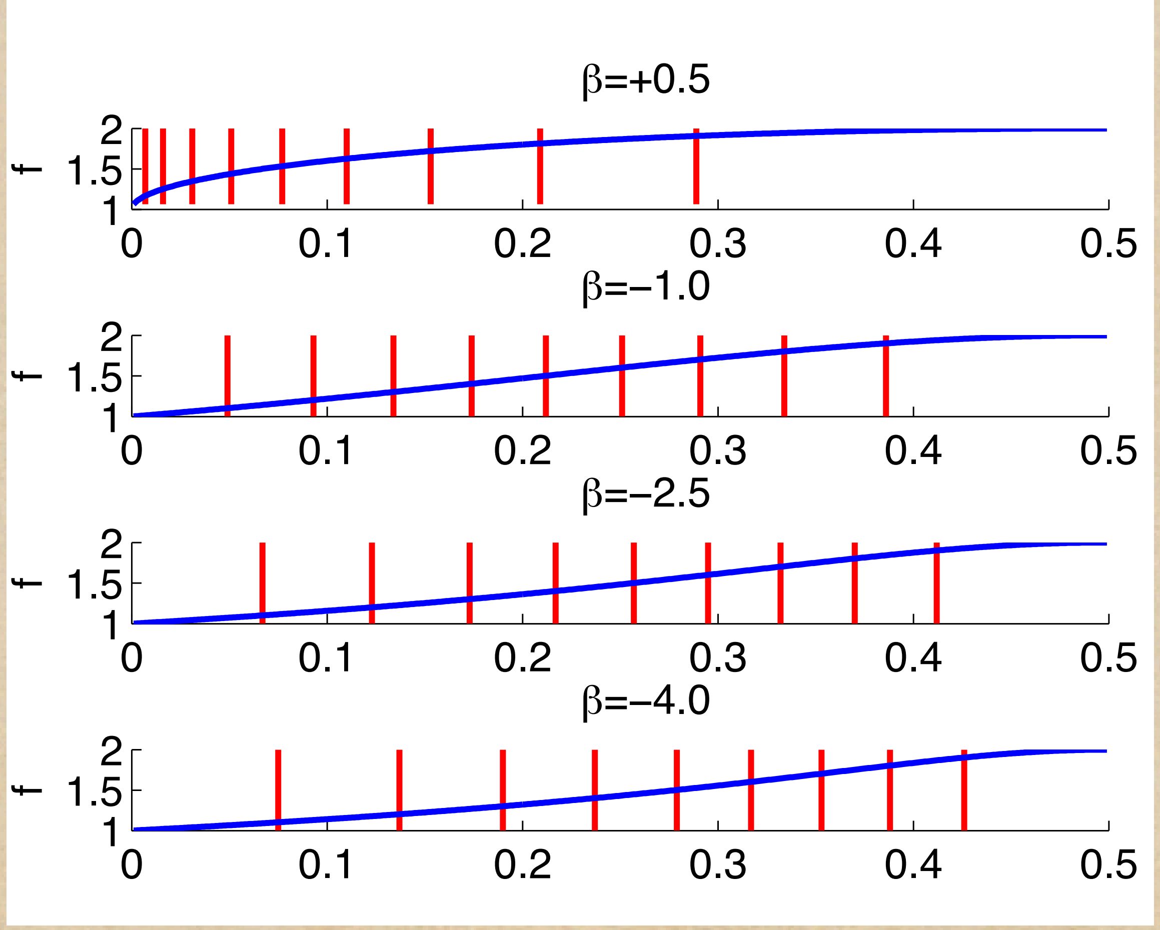
- For any beta<=1,

$$J_\beta(\mathbf{x}) = \frac{1}{n} f_\beta(\mathbf{x})$$

- Properties:
  - Bounded in [0,1]
  - Fairness increases if and only if adding resources to users with resources below a threshold

Useful to Generalize

- Tradeoff between "resolution" and "strictness"



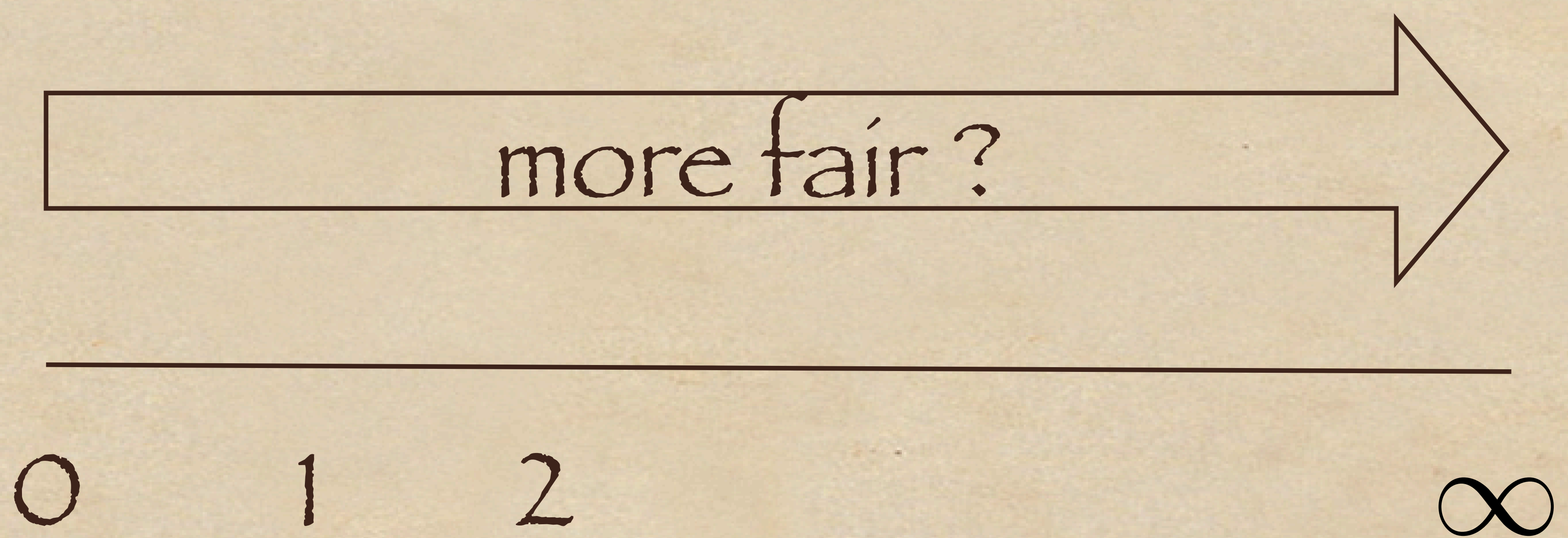
Alpha-Fair Utility

- Factorization of utility function into:
  - Fairness component
  - Efficiency component

$$U_\alpha(x) = \begin{cases} \frac{x^{1-\alpha}}{1-\alpha} & \alpha \geq 0,\ \alpha \neq 1 \\ \log(x) & \alpha = 1 \end{cases}$$

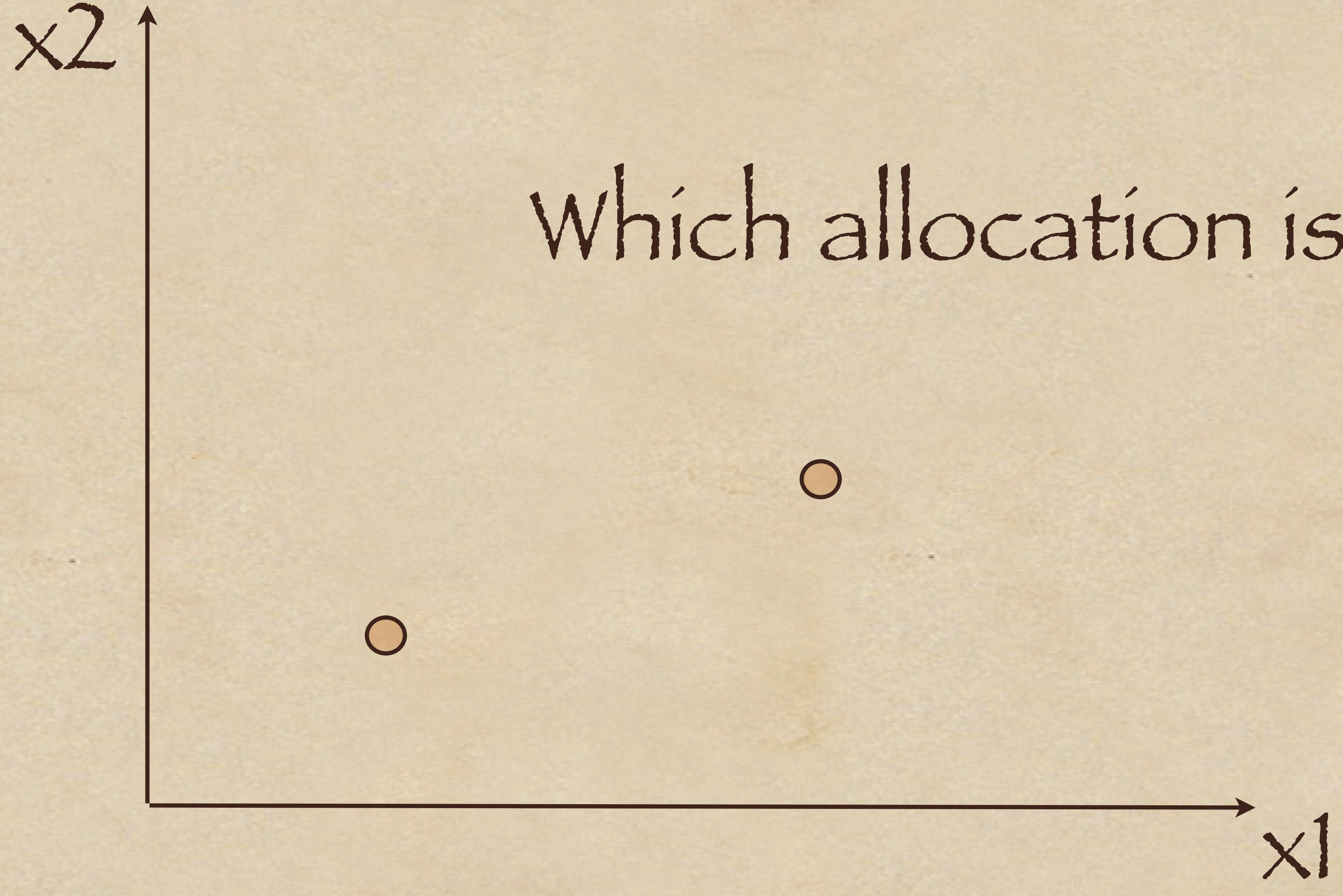
$$= |f_\beta(\mathbf{x})|^\beta \cdot U_\beta\left(\sum_i x_i\right)$$

Alpha Line



Understood for {0,1,infty}, but not [0,infty)

Fair-Efficient Tradeoff



Which allocation is better?

Welfare Function

$$\Phi_\lambda(\mathbf{x}) = \lambda\ell\left(f_\beta\left(\mathbf{x}\right)\right) + \ell\left(\sum_i x_i\right)$$

$$l(y) = \mathrm{sign}(y)\log(|y|)$$

- Necessary and sufficient condition on λ s.t. $\Phi_\lambda(y) > \Phi_\lambda(x)$ if y Pareto dominates x:

$$\lambda \leq \left|\frac{\beta}{1-\beta}\right|$$

Joint Measure

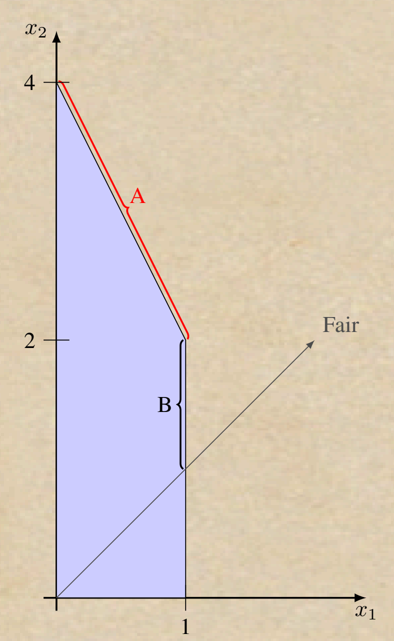
- You pick your definition of "fairness"
- Look at weighted sum of fairness and efficiency
- Can't weigh fairness too much if you want a dominant allocation to be a better allocation
- Plug that "weight threshold" in the joint measure, and you recover alpha-fair utility

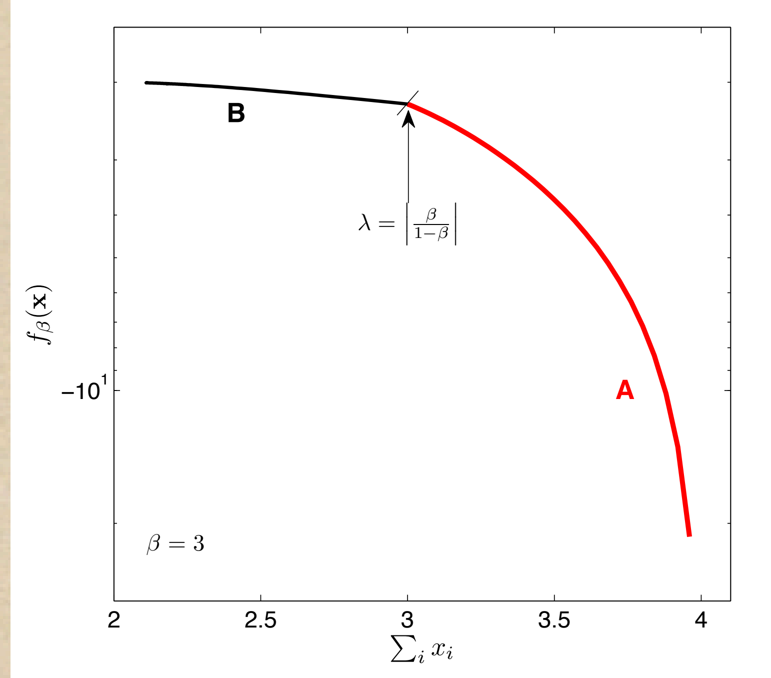
Proof Outline

- Separates into beta<=1 and >1 cases
- Sufficient:
  - Parameterize possible pair of (x,y)
  - Derive sufficient condition
- Necessary:
  - By constructing a contradiction

Larger Alpha More Fair?

- alpha fairness corresponds to the solution of an optimization that places the maximum emphasis on the fairness measure while preserving Pareto efficiency

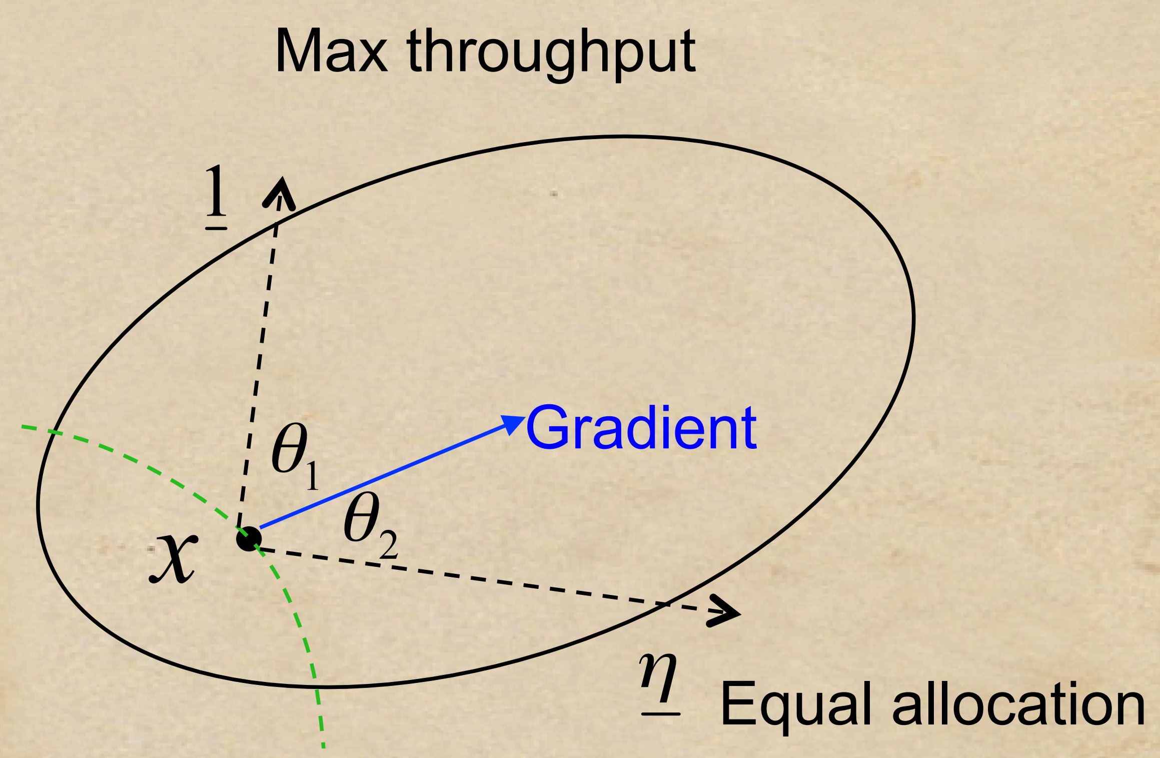




Detour

- Fairness-efficiency reward ratio for a given x:

$$\frac{\left\langle \nabla_{\mathbf{x}} U_{\alpha=\beta}(\mathbf{x}), \frac{\boldsymbol{\eta}}{\|\boldsymbol{\eta}\|} \right\rangle}{\left\langle \nabla_{\mathbf{x}} U_{\alpha=\beta}(\mathbf{x}), \frac{\mathbf{1}_n}{\|\mathbf{1}_n\|} \right\rangle}$$

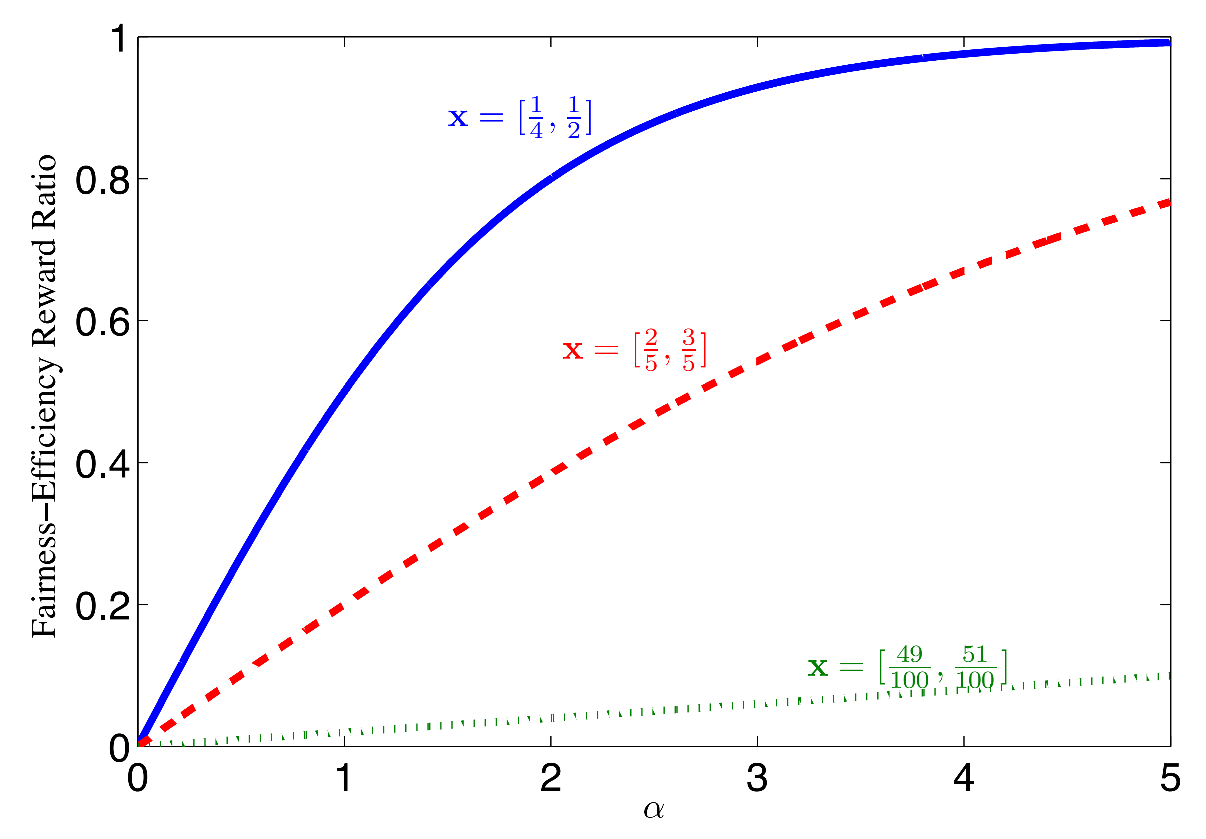


fairness-pointing direction:

$$\eta = \frac{1}{n}\mathbf{1}_n - \frac{\mathbf{x}}{\sum_i x_i}$$

Larger Alpha More Fair?

- Fairness-Efficiency reward ratio increases in alpha

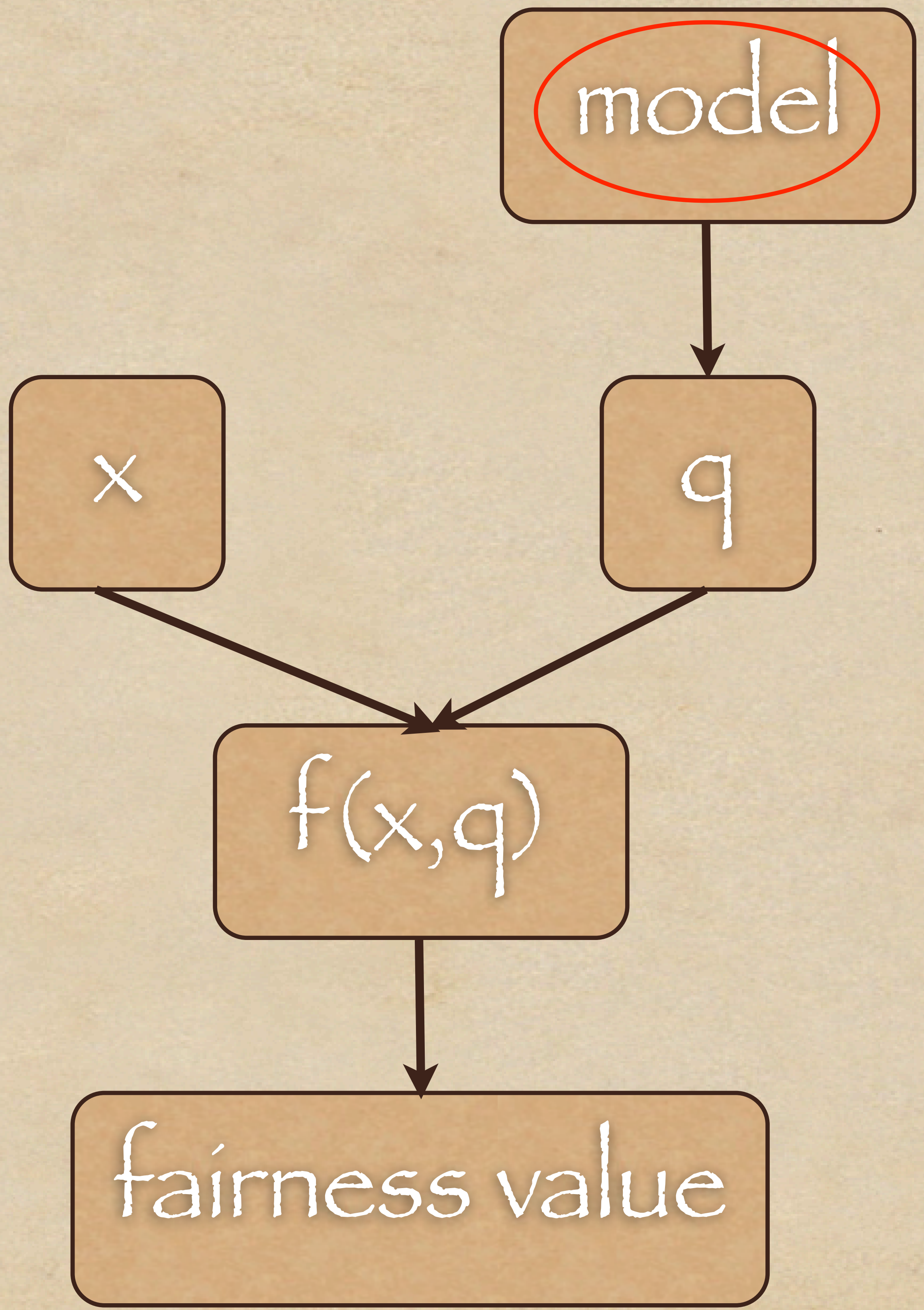


Editing Axioms

- If I don't like a resulting theorem, I need to revisit some of the axioms

Asymmetric Users

- Users are not the same
  - Different valuations of resource
  - Different contributions
- Start with
  - $\mathbf{x}$ resource allocation vector
  - $\mathbf{q}$ user weight vector
- Look for function f mapping into scalar


model
x
q
f(x,q)
fairness value

2nd Set of Axioms

- Same as before except
- Axiom of Partition : mean function weight scaled by user weight

$$s_i = \frac{1}{c} \sum_{k \in i} q_k \left(\sum_{k \in i} x_k \right)^{\rho}$$

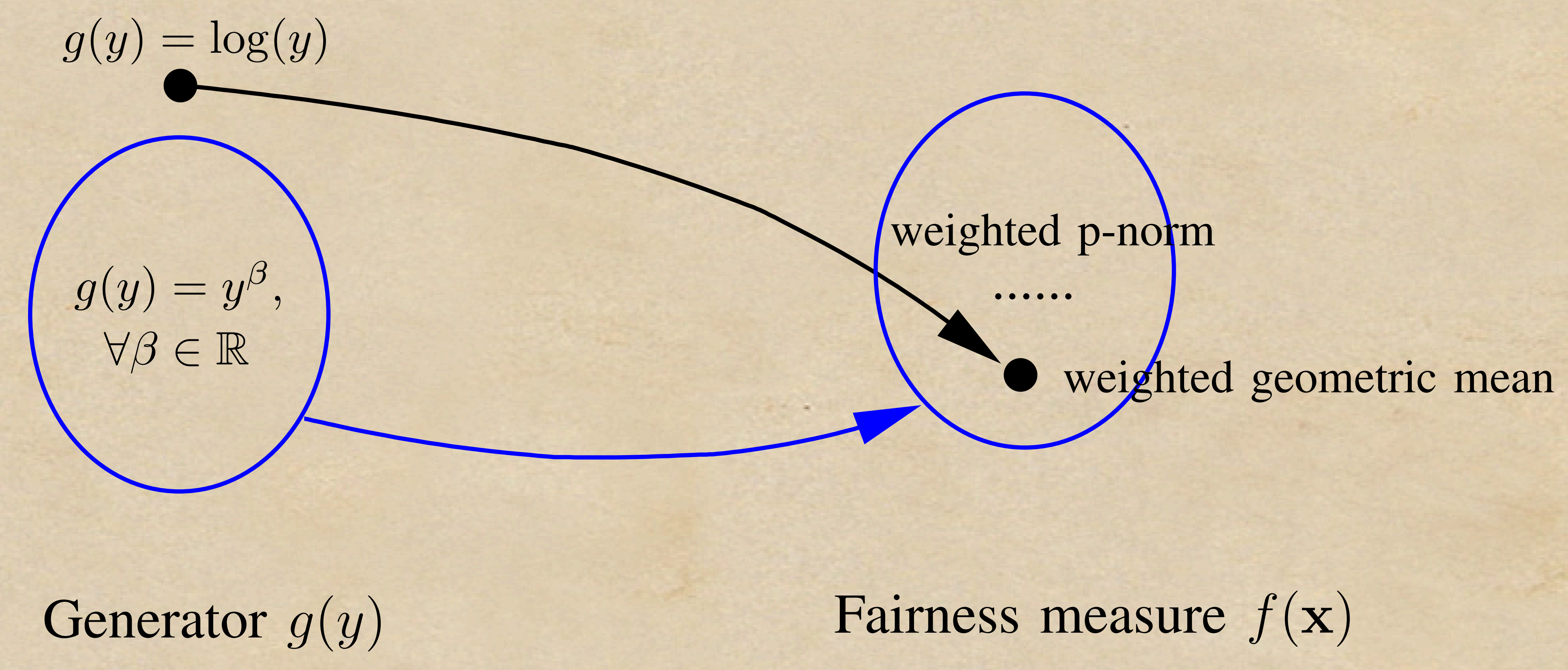
- Axiom of Starvation: under equal weight vectors

Constructed f

- Existence and uniqueness proved
- Unified representation of the new fairness measure

$$f_\beta(\mathbf{x}, \mathbf{q}) = \mathrm{sign}(-1-\beta)\left[\sum_{i=1}^{n} q_i \left(\frac{x_i}{\sum_{j=1}^{n} x_j}\right)^{-\beta}\right]^{\frac{1}{\beta}}$$

What We Recover Now



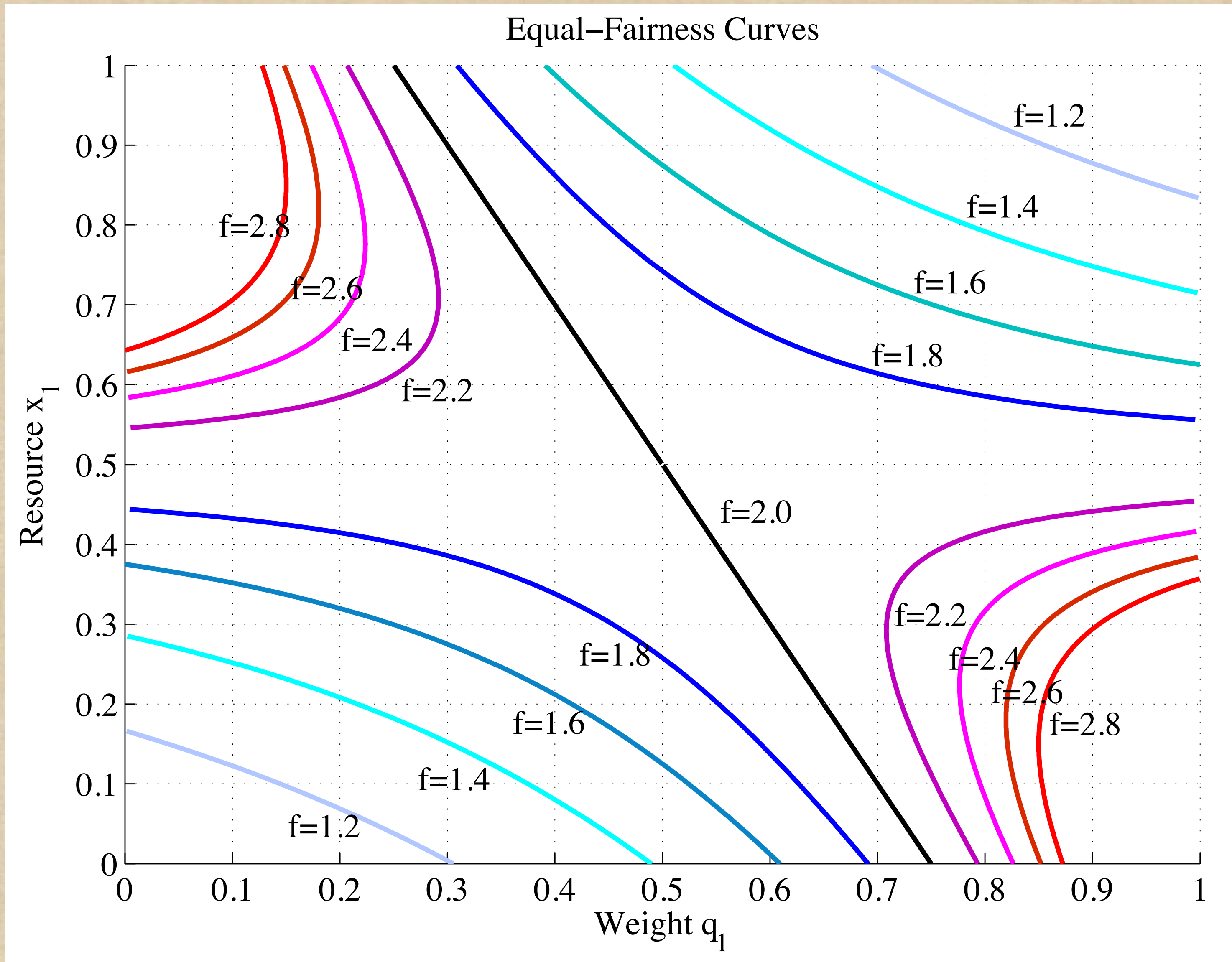
Properties

- Still Schur-concave
- Symmetry and equal-allocation-maximize-fairness no longer true
- Fairness maximizing:

$$x_i = q_i^{\frac{1}{1+\beta}}$$

- Robin-Hood operation tilted as above

Equal-Fairness Curves



3rd Set of Axioms

- Magnitude should matter
- What if we remove Axiom of Homogeneity?
  - Construct fairness measures F that do not decouple efficiency from fairness
  - Generalize earlier results

4 Axioms

- Axiom of Continuity
- Axiom of Saturation
- Axiom of Partition
- Axiom of Starvation

"Final" Form:

$$F_{\beta,\lambda}(\mathbf{x},\mathbf{q}) = f_\beta(\mathbf{x},\mathbf{q}) \left(\sum x_i\right)^{1/\lambda}$$

- f(x) satisfies Original Axioms 1-5
- f(x,q) satisfies 2nd Axioms 1-5
- $1/\lambda$ is degree of homogeneity
- Entire alpha-fair function constructed
- Existence and uniqueness proved

Differences: f or F?

- Equal allocation may not be most fair even for equally-weighted users:
  - F(1,1)<F(0.5,5) for $\beta = 0.5, \lambda = 0.25$
- May not be Schur-concave
- There exists a minimum degree of homogeneity to ensure Pareto efficiency

$$F_{\beta,\lambda}(\mathbf{x},\mathbf{q}) = \operatorname{sign}(-1-\beta)\left[\sum_{i=1}^{n} q_i \left(\frac{x_i}{\sum_{j=1}^{n} x_j}\right)^{-\beta}\right]^{\frac{1}{\beta}} \left(\sum_{i=1}^{n} x_i\right)^{\frac{1}{\lambda}}$$

Three Fairness Measures

$$f_\beta(\mathbf{x})$$

$$f_\beta(\mathbf{x}, \mathbf{q})$$

$$F_{\beta,\lambda}(\mathbf{x}, \mathbf{q})$$

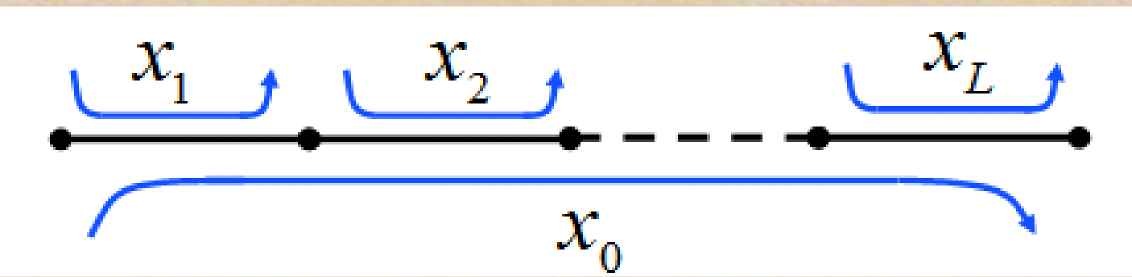
Reverse Engineering f

- Which is more fair?
  - (1,1)
  - (x,x+c)
- Let's try the experiment now

Illustrations

- Examples from communication networks

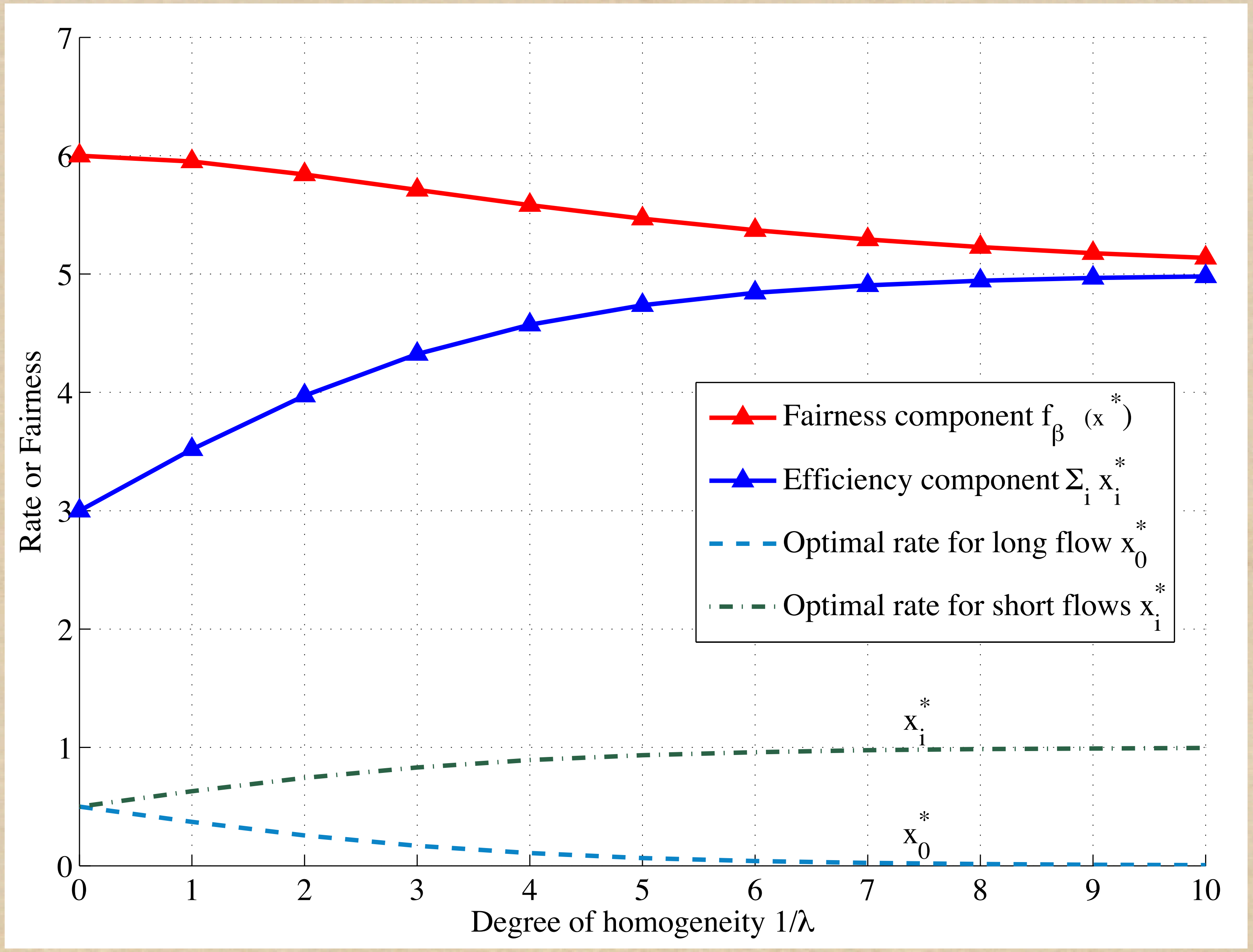
Congestion Control



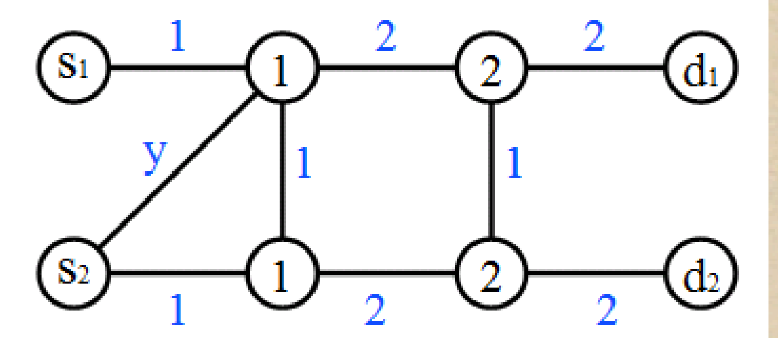
$$\max_{\mathbf{x}} \quad F_{\beta,\lambda}(\mathbf{x}) = f_\beta(\mathbf{x}) \cdot \left(\sum_i x_i\right)^{\frac{1}{\lambda}}$$
$$\text{s.t.} \quad x_0 + x_i \le 1, \ \text{for } i = 1, \ldots, L.$$

Generalize alpha-fair utility objective

Congestion Control



Routing

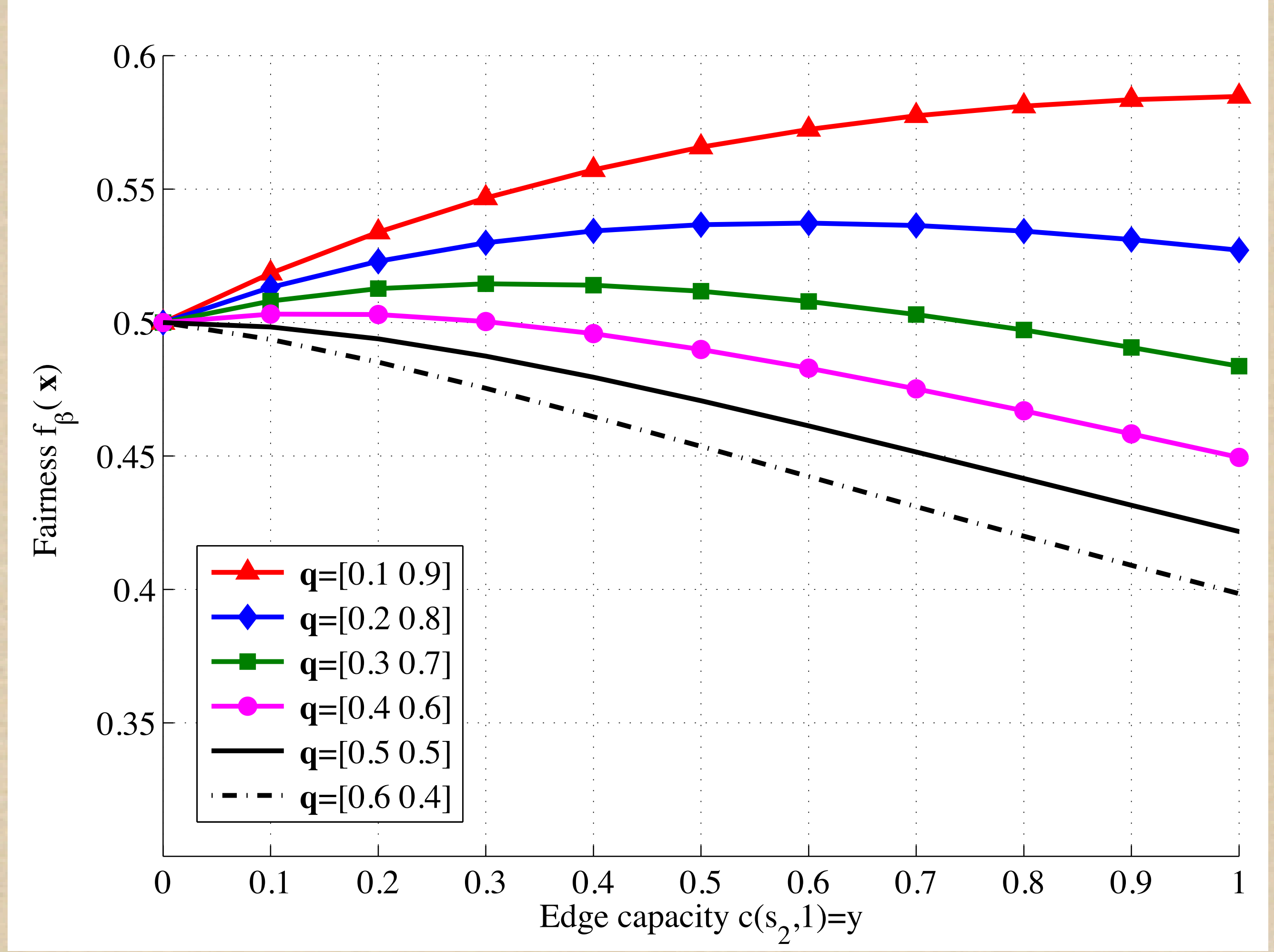


Consider 2-commodity flow routing

When will larger y lead to more fairness?

If user 2 is weighed heavy enough

Routing



$$\frac{q_2}{q_1} \geq (1 + y_{max})^{1+\beta}$$

Power Control

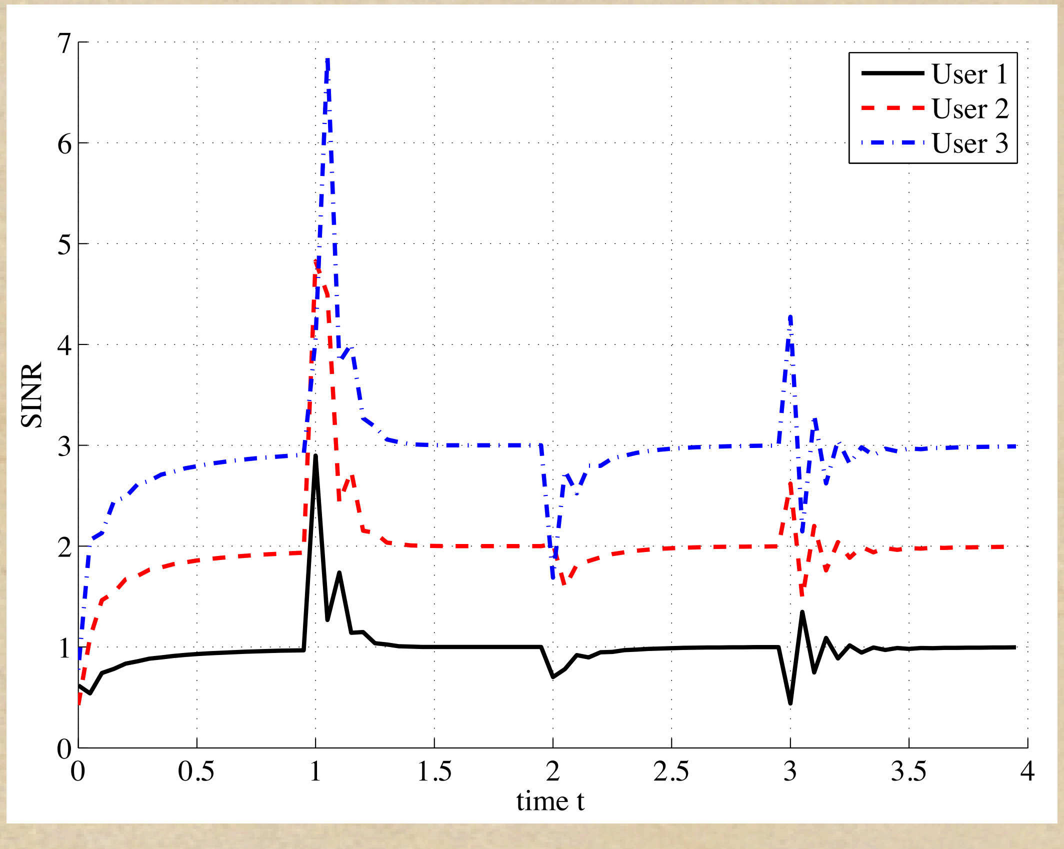
- Signal-Interference-Ratio:

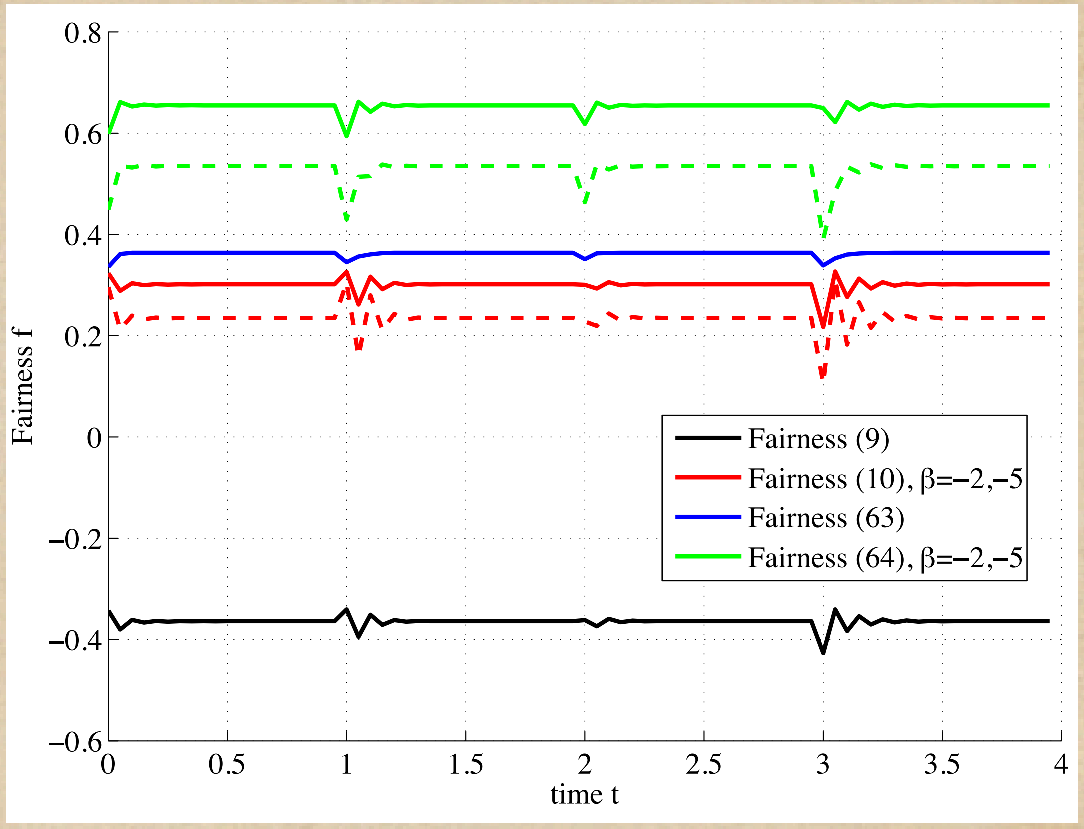
$$\gamma_i = \frac{G_{ii} \cdot p_i}{\sum_{j \neq i} G_{ij} \cdot p_j + \sigma^2}$$

- Foschini-Miljanic Power Control:

$$p_i[k] = \frac{\gamma_i^*}{\gamma[k-1]} p_i[k-1]$$

Power Control

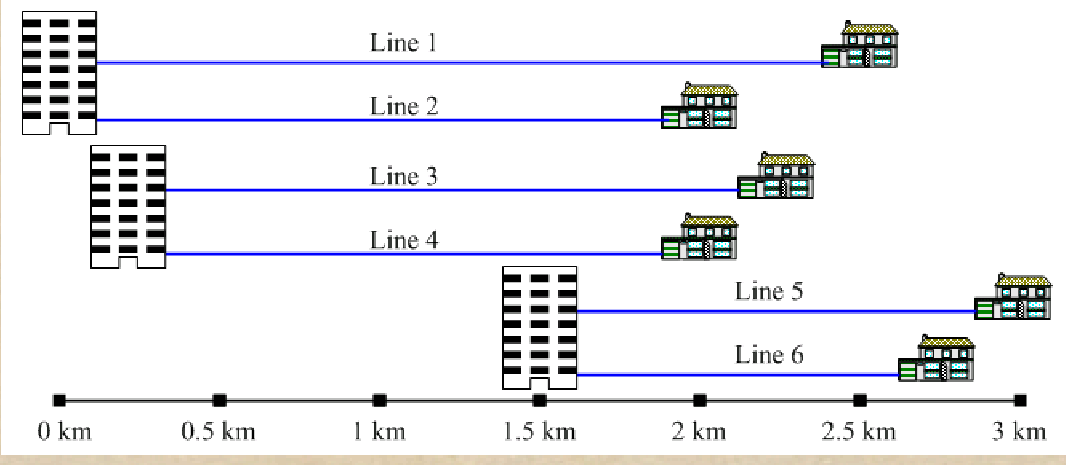




Smaller beta is stricter

fairness value more sensitive

Spectrum Management



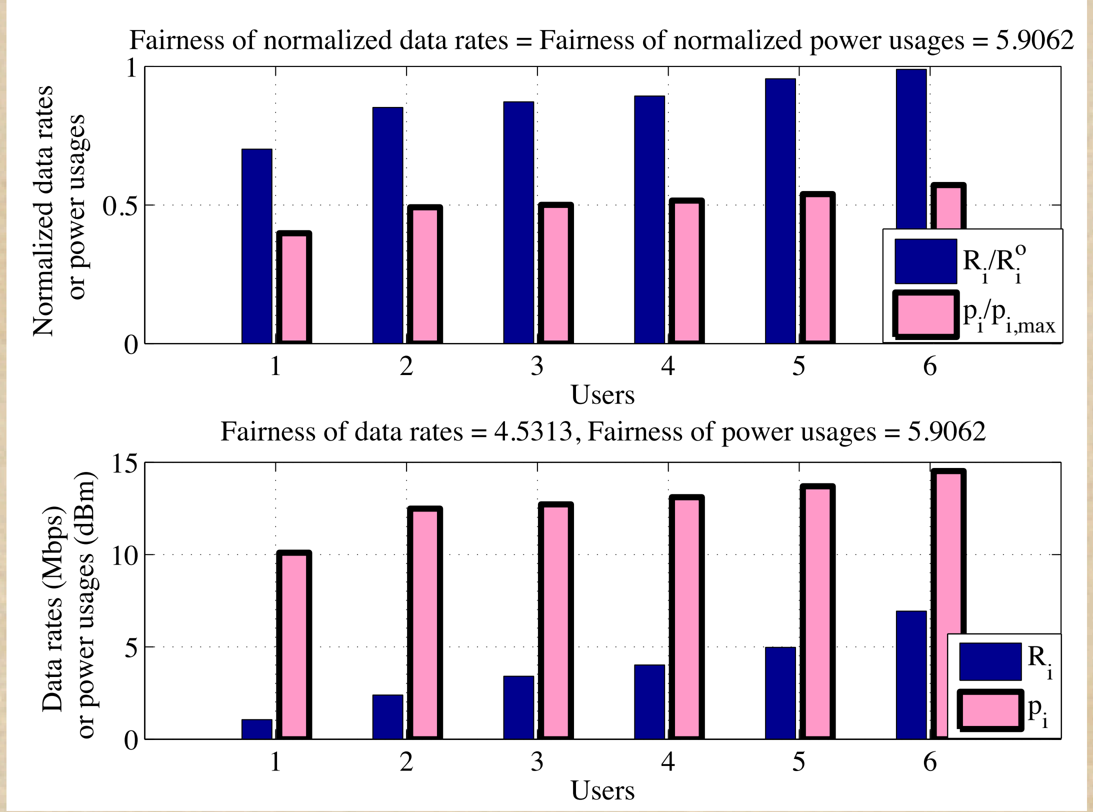
$$\max_{\{p_i^k, R_i\}} \quad \sum_{i=1}^{n} R_i \qquad\qquad ($$

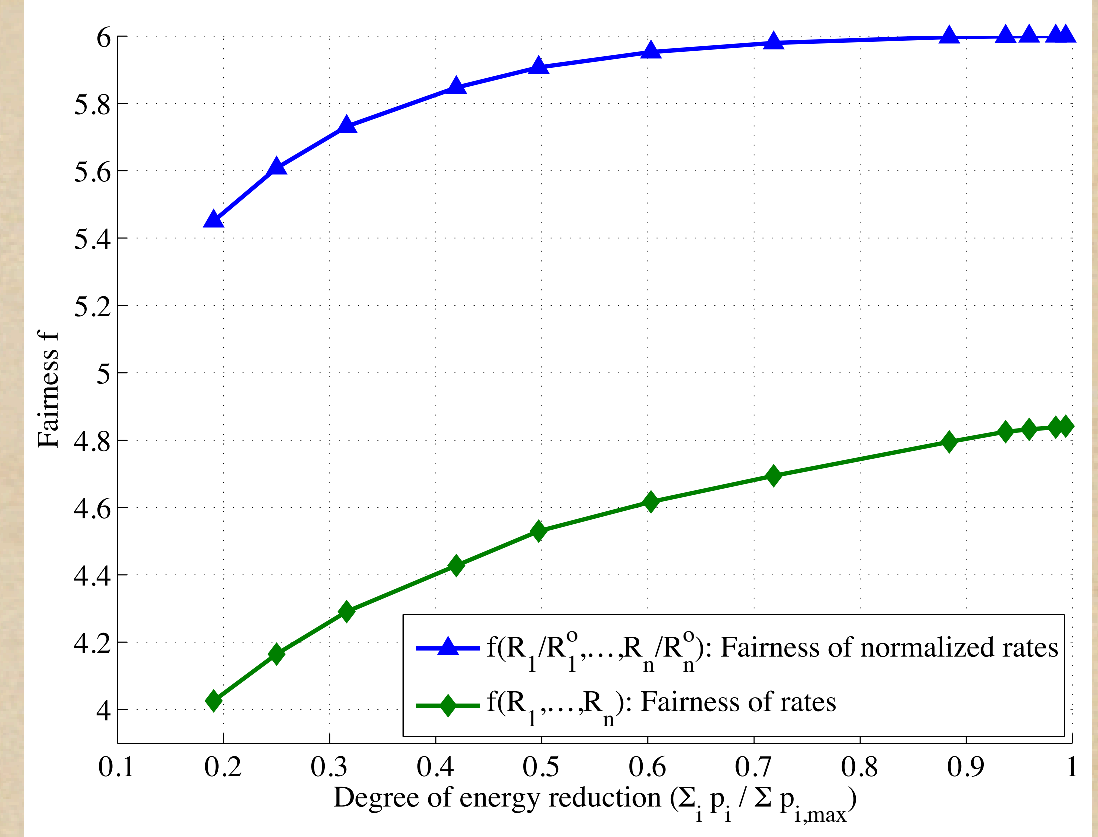
$$\text{s.t.} \quad p_i = \sum_{k=1}^{K} p_i^k \leq p_{i,\max}, \quad \text{for } i = 1, \ldots, n$$

$$R_i = \eta \cdot \sum_{k=1}^{K} \psi_i^k(\mathbf{G}^k, \mathbf{p}^k), \quad \text{for } i = 1, \ldots, r$$

$$\frac{R_i}{R_i^o} / \frac{p_i}{p_{i,\max}} = \rho, \quad \text{for } i = 1, \ldots, n$$

Spectrum Management





Allocating "price" of energy reduction

What's Here?

- Axiomatic construction, 1-2 parameters
- Uniqueness
- Continuous (rather than binary) quantification
- Unification (of decomposable measures)
- Tunable fairness-efficiency tradeoff
- Consistency check
- Connecting with other fields

What's Missing?

- Discrete/multi-resource/multi-aggregation Allocation
- Time-dependent valuation and time evolution
- User-specific valuation and procedure

Not the End Yet

- What have others done (many years ago and from different fields)?
- What has not been done?

THE BARGAINING PROBLEM[1]

By John F. Nash, Jr.

A new treatment is presented of a classical economic problem, one which occurs in many forms, as bargaining, bilateral monopoly, etc. It may also be regarded as a nonzero-sum two-person game. In this treatment a few general assumptions are made concerning the behavior of a single individual and of a group of two individuals in certain economic environments. From these, the solution (in the sense of this paper) of the classical problem may be obtained. In the terms of game theory, value found for the game.

A VALUE FOR n-PERSON GAMES[1]

L. S. Shapley

§1. Introduction

At the foundation of the theory of games is the assumption that the players of a game can evaluate, in their utility scales, every "prospect" that might arise as a result of a play. In attempting to apply the theory to any field, one would normally expect to be permitted to include in the class of "prospects," the prospect of having to play a game. The possibility of evaluating games is therefore of critical importance. So long as the theory is unable to assign values to the games typically found in application, only relatively simple situations—where games do not depend on other games—will be susceptible to analysis and solution.

In the finite theory of von Neumann and Morgenstern[2] difficulty in evaluation persists for the "essential" games, and for only those. In this note we deduce a value for the "essential" case and examine a number of its elementary properties. We proceed from a set of three axioms, having simple intuitive interpretations, which suffice to determine the value uniquely.

Our present work, though mathematically self-contained, is founded conceptually on the von Neumann-Morgenstern theory up to their introduction of characteristic functions. We thereby inherit certain mportant underlying assumptions: (a) that utility is objective and trans- rable; (b) that games are cooperative affairs; (c) that games, granting and (b), are adequately represented by their characteristic functions. vever, we are not committed to the assumptions regarding rational vior embodied in the von Neumann-Morgenstern notion of "solu-

ON MEASURES OF ENTROPY AND INFORMATION

ALFRÉD RÉNYI
MATHEMATICAL INSTITUTE
HUNGARIAN ACADEMY OF SCIENCES

1. Characterization of Shannon's measure of entropy

Let $\mathcal{P} = (p_1, p_2, \cdots, p_n)$ be a finite discrete probability distribution, that is, suppose $p_k \geqq 0 (k = 1, 2, \cdots, n)$ and $\sum_{k=1}^{n} p_k = 1$. The amount of uncertainty of the distribution $\mathcal{P}$, that is, the amount of uncertainty concerning the outcome of an experiment, the possible results of which have the probabilities $p_1, p_2, \cdots, p_n$, is called the *entropy* of the distribution $\mathcal{P}$ and is usually measured by the quantity $H[\mathcal{P}] = H(p_1, p_2, \cdots, p_n)$, introduced by Shannon [1] and defined by

$$H(p_1, p_2, \cdots, p_n) = \sum_{k=1}^{n} p_k \log_2 \frac{1}{p_k}. \quad (1.1)$$

Different sets of postulates have been given, which characterize the quantity (1.1). The simplest such set of postulates is that given by Fadeev [2] (see also Feinstein [3]). Fadeev's postulates are as follows.

(a) $H(p_1, p_2, \cdots, p_n)$ *is a symmetric function of its variables for* $n = 2, 3, \cdots$.
(b) $H(p, 1 - p)$ *is a continuous function of* p *for* $0 \leqq p \leqq 1$.
(c) $H(1/2, 1/2) = 1$.
(d) $H[tp_1, (1 - t)p_1, p_2, \cdots, p_n] = H(p_1, p_2, \cdots, p_n) + p_1 H(t, 1 - t)$ *for any distribution* $\mathcal{P} = (p_1, p_2, \cdots, p_n)$ *and for* $0 \leqq t \leqq 1$.

Axiomatic Constructions

- Renyi, Shannon
- Atkinson
- Mean (Aczel)
- Gini and stochastic dominance (Aaberge)
- Nash
- Shapley, Myerson, Raiffa
- Expected utility (von Neumann-Morgenstern)
- Social welfare (Bergson-Samuelson, Harsanyi)

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global, decomposable

global

local closedness

Statistics and CS Theory

Here starts the speculative part

Renyi Entropy H (1960)

- Continuity
- Symmetry
- Additivity
- Mean-value property
- Normalization $H(0.5)=1$

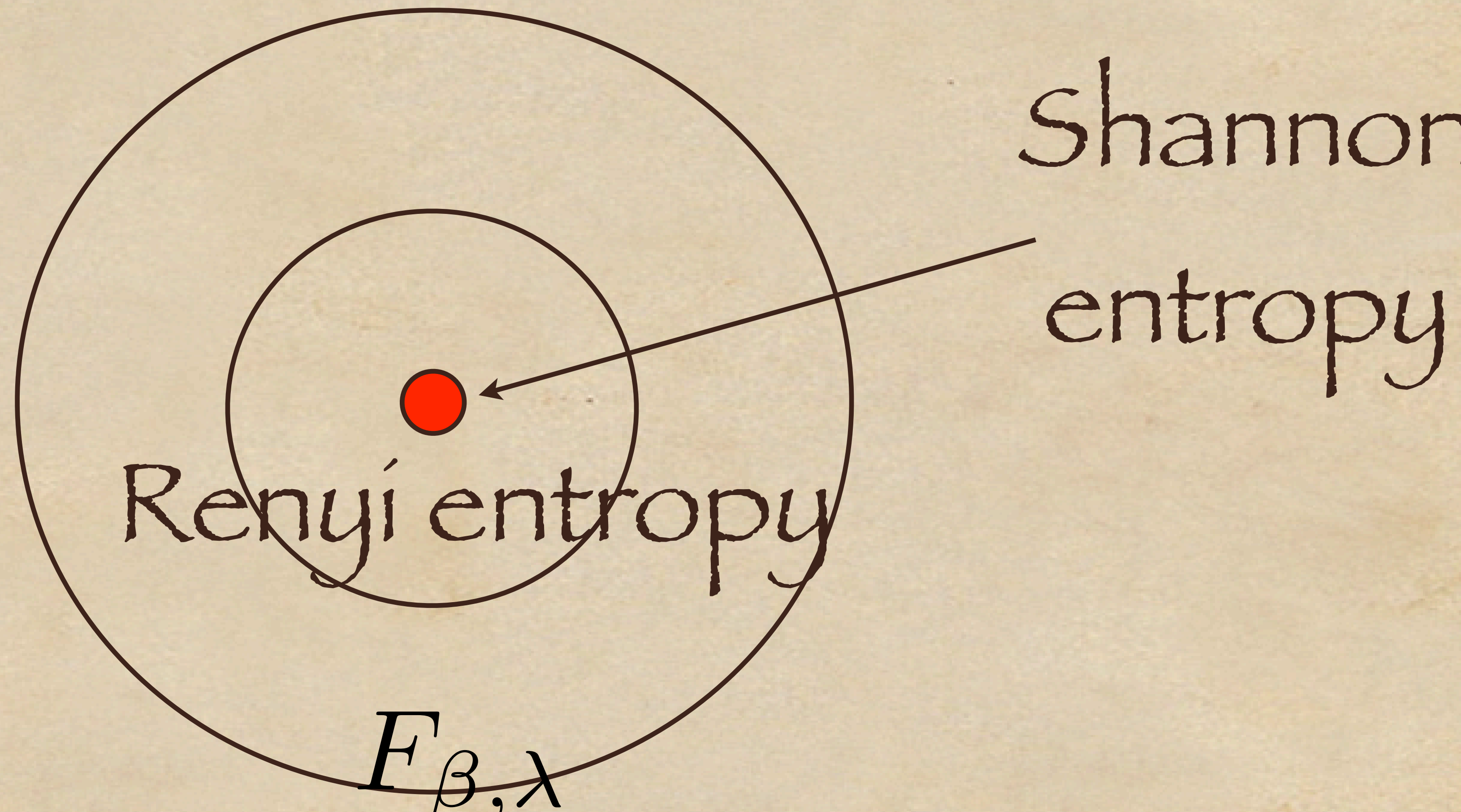
Comparison of Axioms

- Continuity : Continuity
- Normalization: Homogeneity
- Additivity + Mean-value: Partition
- ? : Starvation and Saturation

Renyi Entropy H

- Shannon entropy is a special limiting case
- For beta<=1, F is exponentiated Renyi entropy by $\beta = 1 - \alpha, \lambda = -1$
    - Less varied: more certain and more fair
- For beta>1: generalization of Renyi entropy
    - Turns out to be alpha fairness
- Beta between 0 and 1 is the fun part

Recap Relationships



Fairness as No-Envy

- Ultimatum Game
- Cake-Cutting
  - Efficient: Pareto optimality
  - Fair/Stable: No-envy
  - Strategy-proof?
- Fair-Division

Economics

Expected Utility Theory

von Neumann-Morgenstern 1944

- Completeness
- Transitivity
- Continuity
- Independence

NBS and Shapley Value

- Symmetry
- Affine invariance
- IIA
- Pareto optimality

- Symmetry
- Additivity
- Dummy
- Efficiency

User Reaction Model

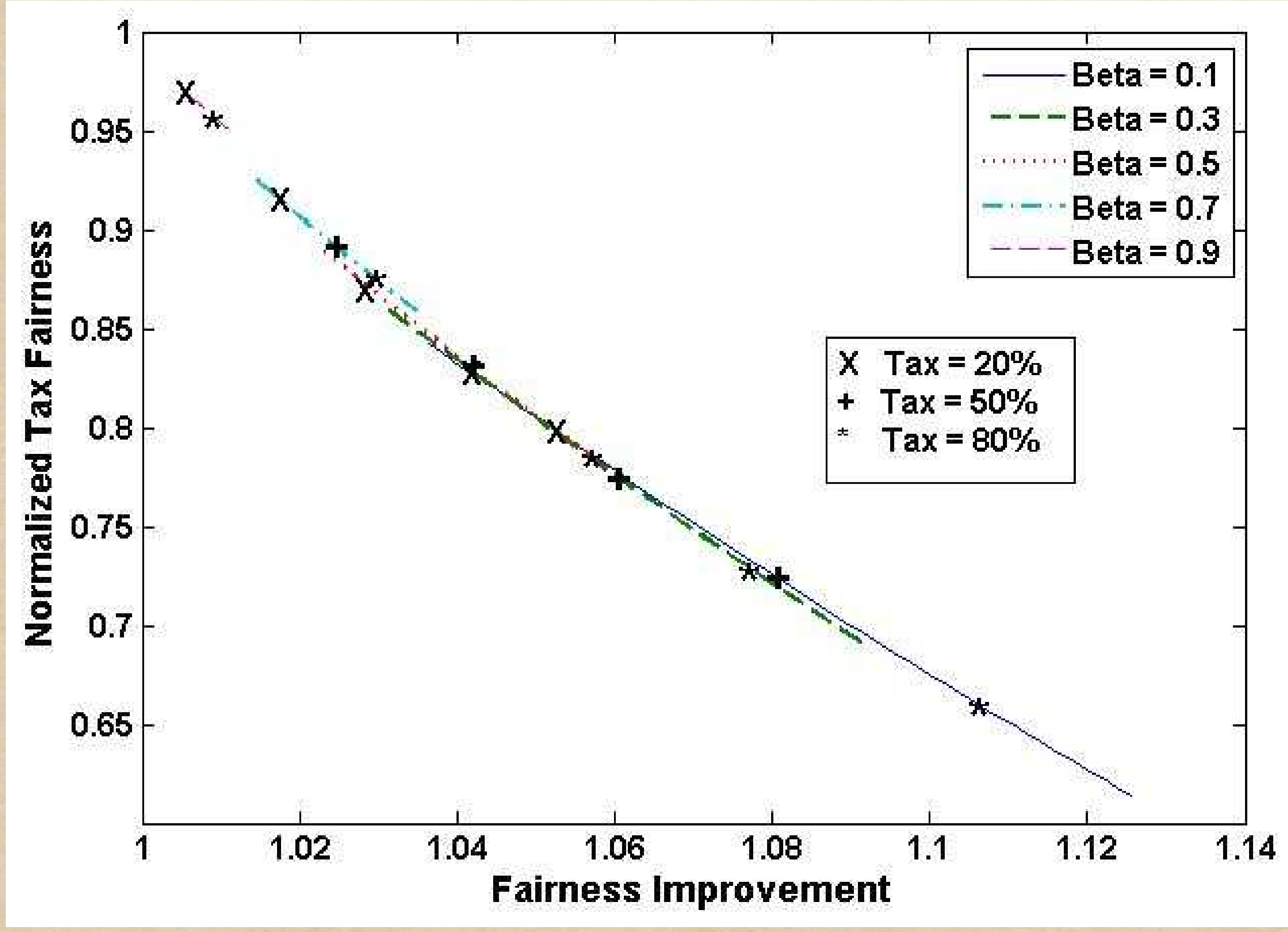
- Alesina and Angeletos 2005 2006
- Stability, long-term fairness and efficiency under F?
- Optimal taxation theory
  - Marginal tax schedule for F?

Fairness of Taxation

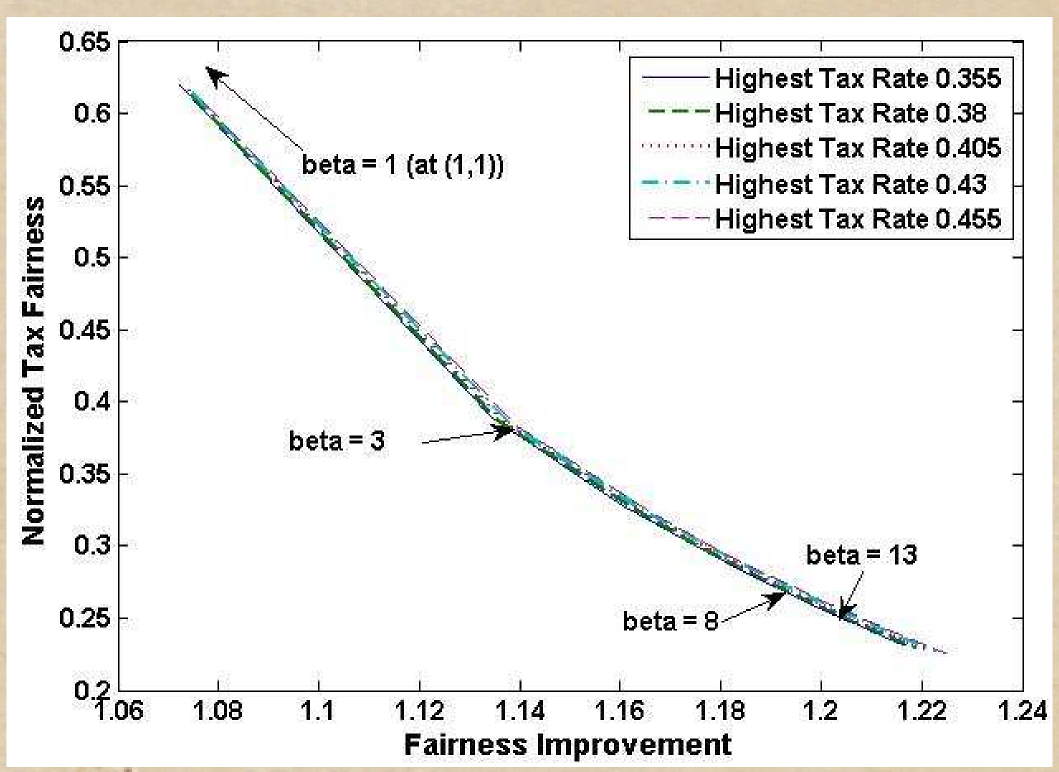
Tax rate (%)	Single Bracket ($)	Married Bracket ($)
10	0-8375	0-16750
15	8376-34000	16751-68000
25	34001-82400	68001-137300
28	82401-171850	137301-209250
33	171851-373650	209251-373650
35	373650+	373650+

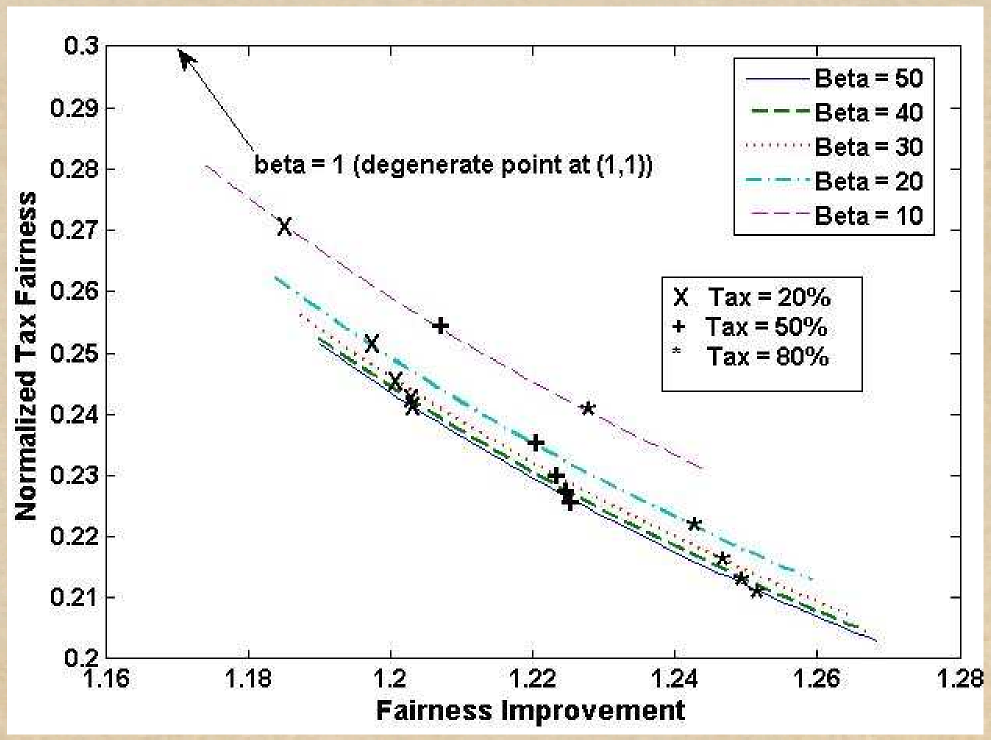
Bracket	Users
0-4000	1
4000-8375	24
8376-34000	38
34001-82400	30
82401-171850	20
171851-373650	42
373651+	1
Total	50

Fair Tax

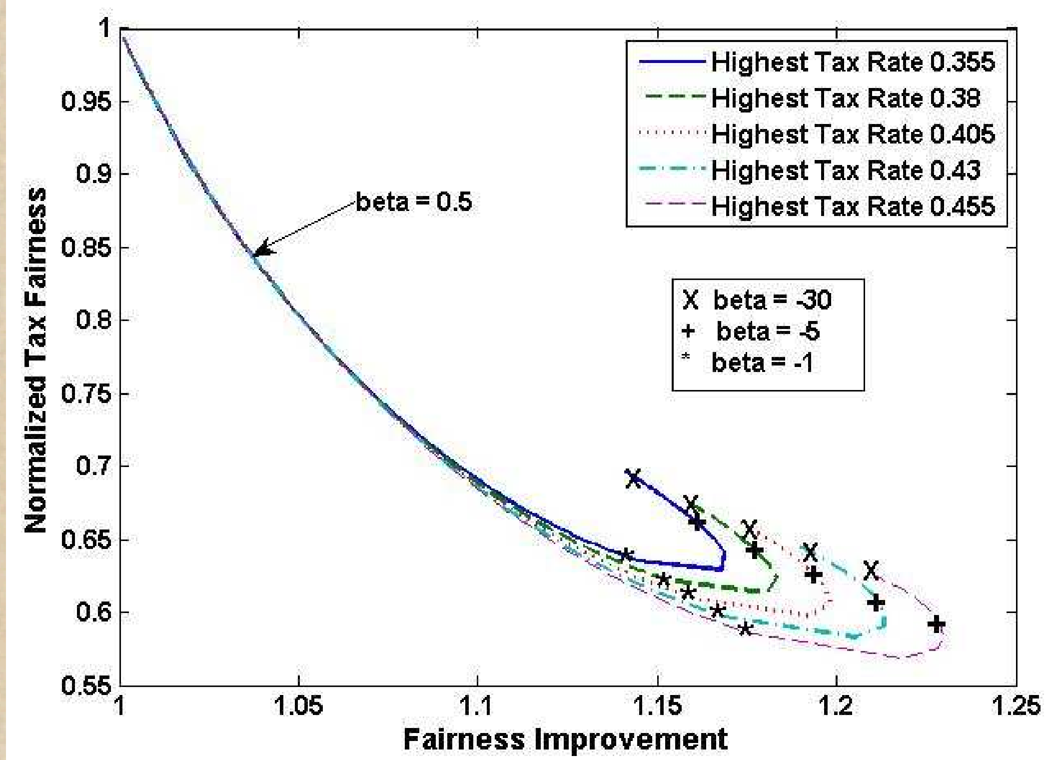

Normalized Tax Fairness
Fairness Improvement
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
1
1.02
1.04
1.06
1.08
1.1
1.12
1.14
Beta = 0.1
Beta = 0.3
Beta = 0.5
Beta = 0.7
Beta = 0.9
X Tax = 20%
+ Tax = 50%
* Tax = 80%

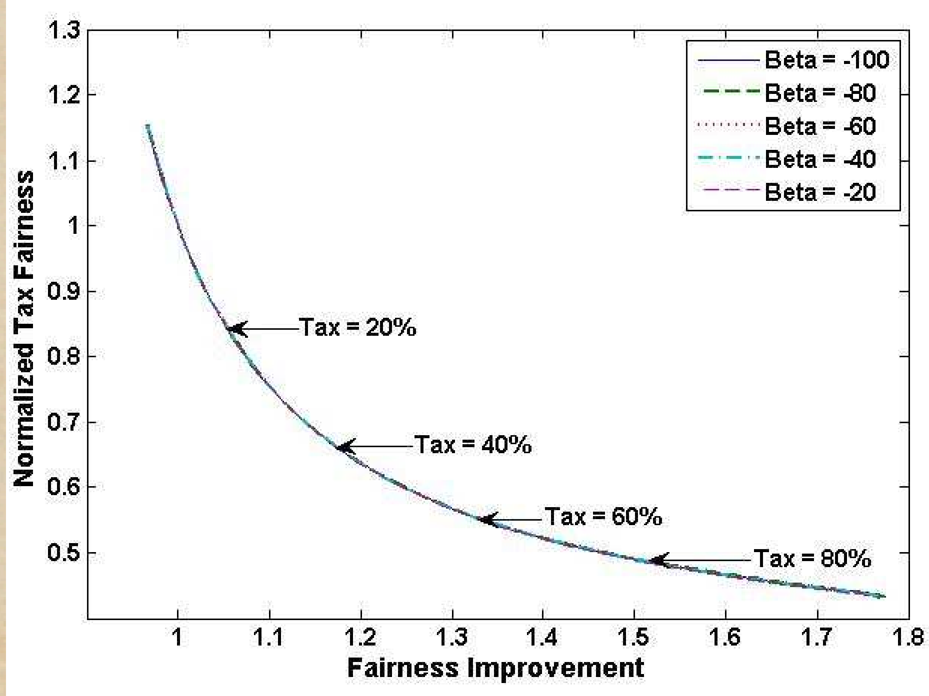
Fair Tax


Highest Tax Rate 0.355
Highest Tax Rate 0.38
Highest Tax Rate 0.405
Highest Tax Rate 0.43
Highest Tax Rate 0.455
beta = 1 (at (1,1))
beta = 3
beta = 8
beta = 13
Normalized Tax Fairness
Fairness Improvement


Beta = 50
Beta = 40
Beta = 30
Beta = 20
Beta = 10
X Tax = 20%
+ Tax = 50%
* Tax = 80%
beta = 1 (degenerate point at (1,1))
Normalized Tax Fairness
Fairness Improvement

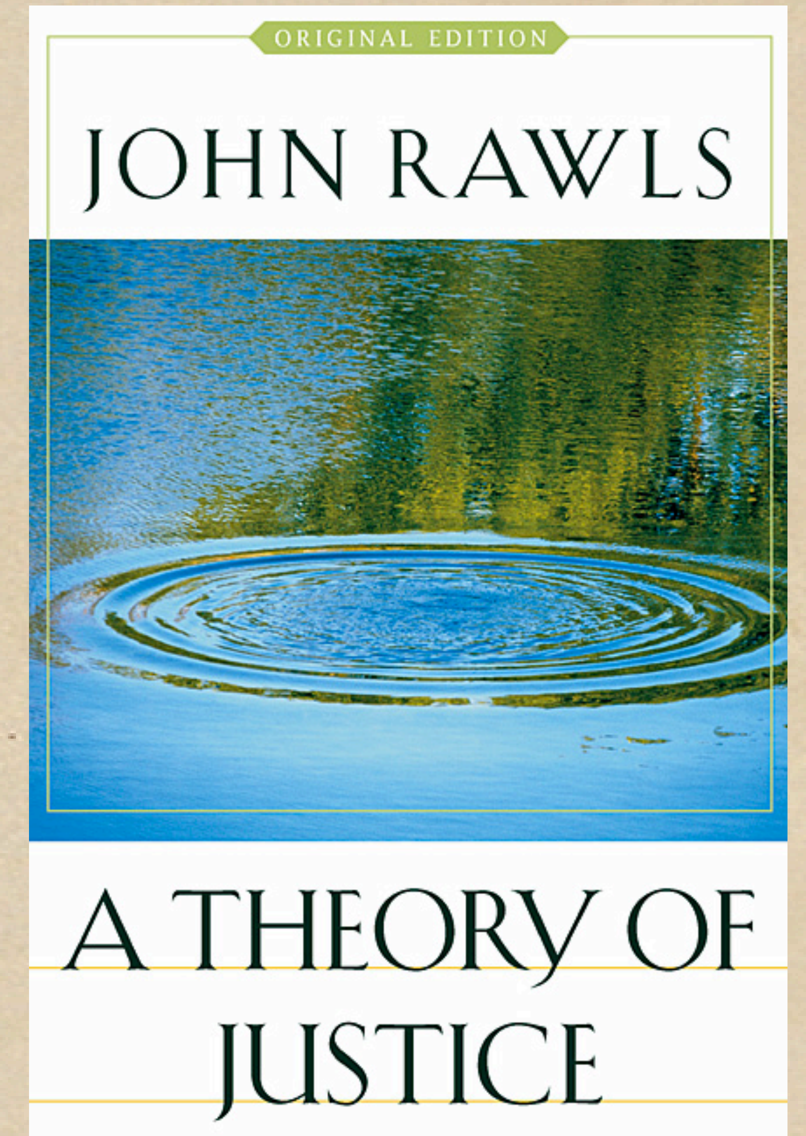
Fair Tax

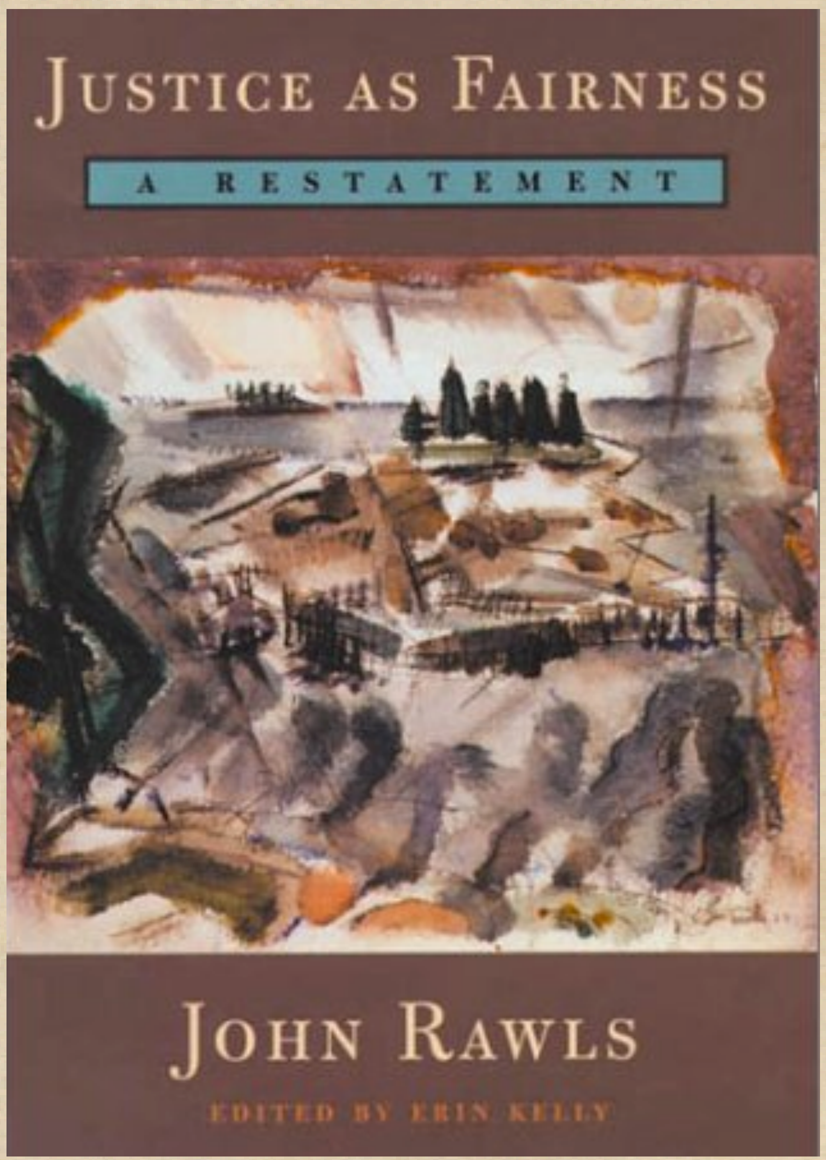

Highest Tax Rate 0.355
Highest Tax Rate 0.38
Highest Tax Rate 0.405
Highest Tax Rate 0.43
Highest Tax Rate 0.455
beta = 0.5
X beta = -30
+ beta = -5
* beta = -1
Normalized Tax Fairness
Fairness Improvement


Beta = -100
Beta = -80
Beta = -60
Beta = -40
Beta = -20
Tax = 20%
Tax = 40%
Tax = 60%
Tax = 80%
Normalized Tax Fairness
Fairness Improvement

Political Philosophy

Principles of Fairness (1971)





Detour: Back in 1950...

- Nash published his dissertation
- Shapley worked on his dissertation
- Rawls started working on justice as fairness after his dissertation

- ...all at Princeton

Rawl's Theory of Justice

- The original position and veil of ignorance
- Principle 1: Each person is to have an equal right to the most extensive scheme of basic liberties compatible with a similar scheme of liberties for others
- Principle 2: Positions open to all with fair equality of opportunity. Inequalities are to be arranged so that they are to be of the greatest benefit to the least-advantaged members

Distributive Justice...

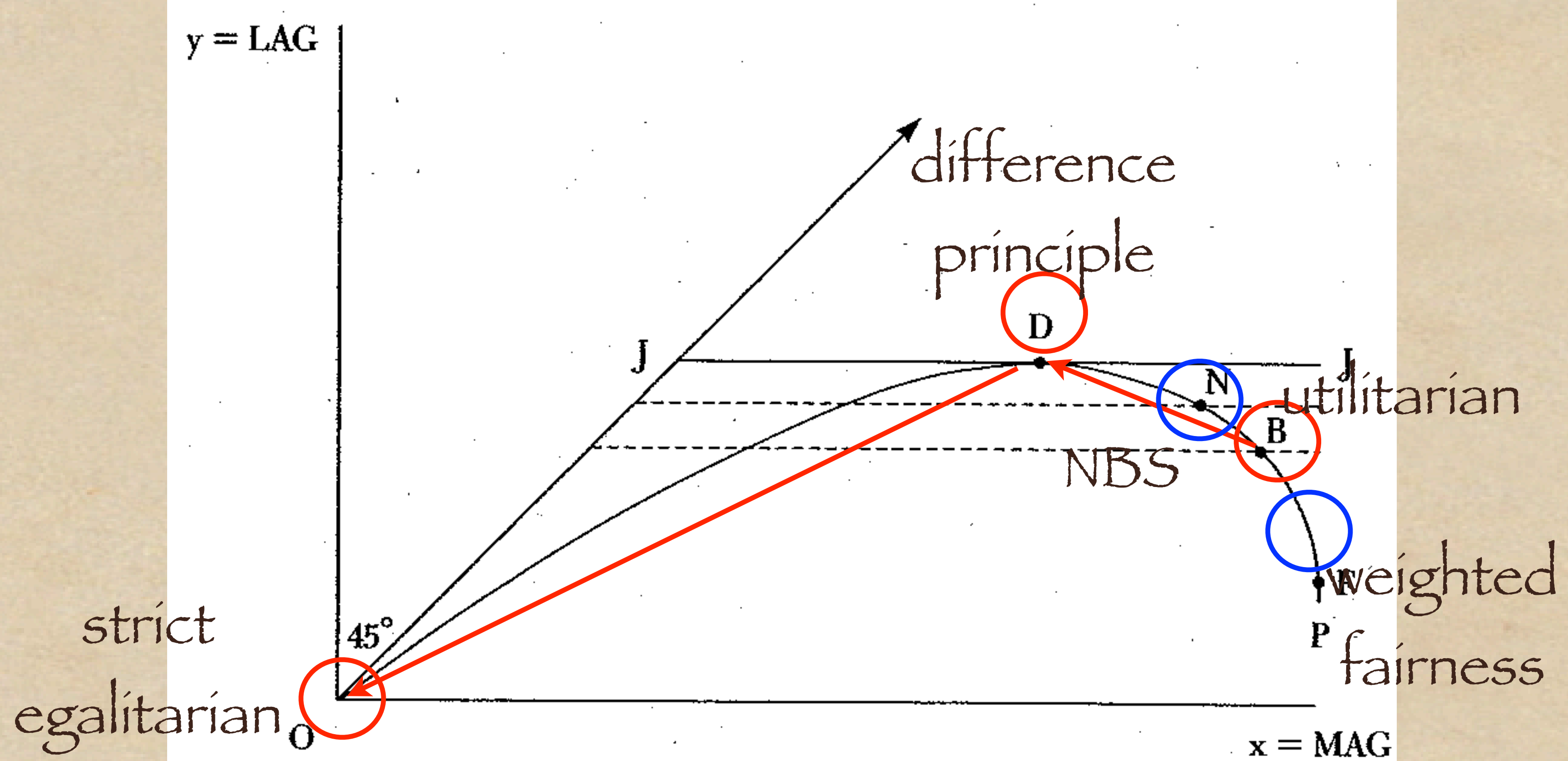
- Axiomatic theories of fairness can be viewed as axiomatic quantification of distributive justice
- Principle 1:
  - constructs the largest possible constraint set
  - quantified by "constant tax hurts fairness" theorem (when will this fail?)

… As Fairness

- Principle 2:
    - Everybody's q could be large
    - Max-min fairness, can be generalized
- Compare 3 points:
    - Fairness maximizing
    - Nash bargaining
    - utility maximizing

Rawls 2001, p.62

Figure 1



In this figure the distances along the two axes are measured in terms of an index of primary goods, with the x-axis the more advantaged group (MAG), the y-axis the less advantaged (LAG). The line JJ parallel to the x-axis is the highest equal-justice line touched by the OP curve at its maximum at D. Note that D is the efficient point nearest to equality, represented by the 45-degree line. N is the Nash point, where the product of utilities is maximized (if we assume utilities to be linear in indexes of primary goods), and B is the Bentham point, where the sum of individual utilities is maximized (again with the same assumption). The set of efficient points goes from D to the feudal point F, at which the OP curve becomes vertical.

Reflective Equilibrium

- Experiment Design
  - 2-user, user 1 works twice as fast
  - allocate income [x1,x2]
  - Which is more fair: [x1,x2], or [y1,y2]?
  - Based on answer, estimate $f_\beta(\mathbf{x}, \mathbf{q})$
- User-interface just designed
  - 4G Fairness Tool with Telcordia

Social Welfare Theory

- Aggregating individual measures to system-wide measure

$$\{f_i(\mathbf{x})\} \rightarrow W(f_1, f_2, \ldots, f_n) \longleftrightarrow f(\mathbf{x})$$

$$\{f_i(x_i)\} \rightarrow W(f_1, f_2, \ldots, f_n)$$

Arrow's impossibility theorem

Kolm theorem

Pigou-Dalton principle

Sociology and Psychology

Atkinson Inequality Index

- Symmetry
- Equal allocation is least unequal
- Robin Hood operation reduces inequality
- Homogeneity
- Population replication has no effect
- Decomposition (by arithmetic mean)

Inequality Index

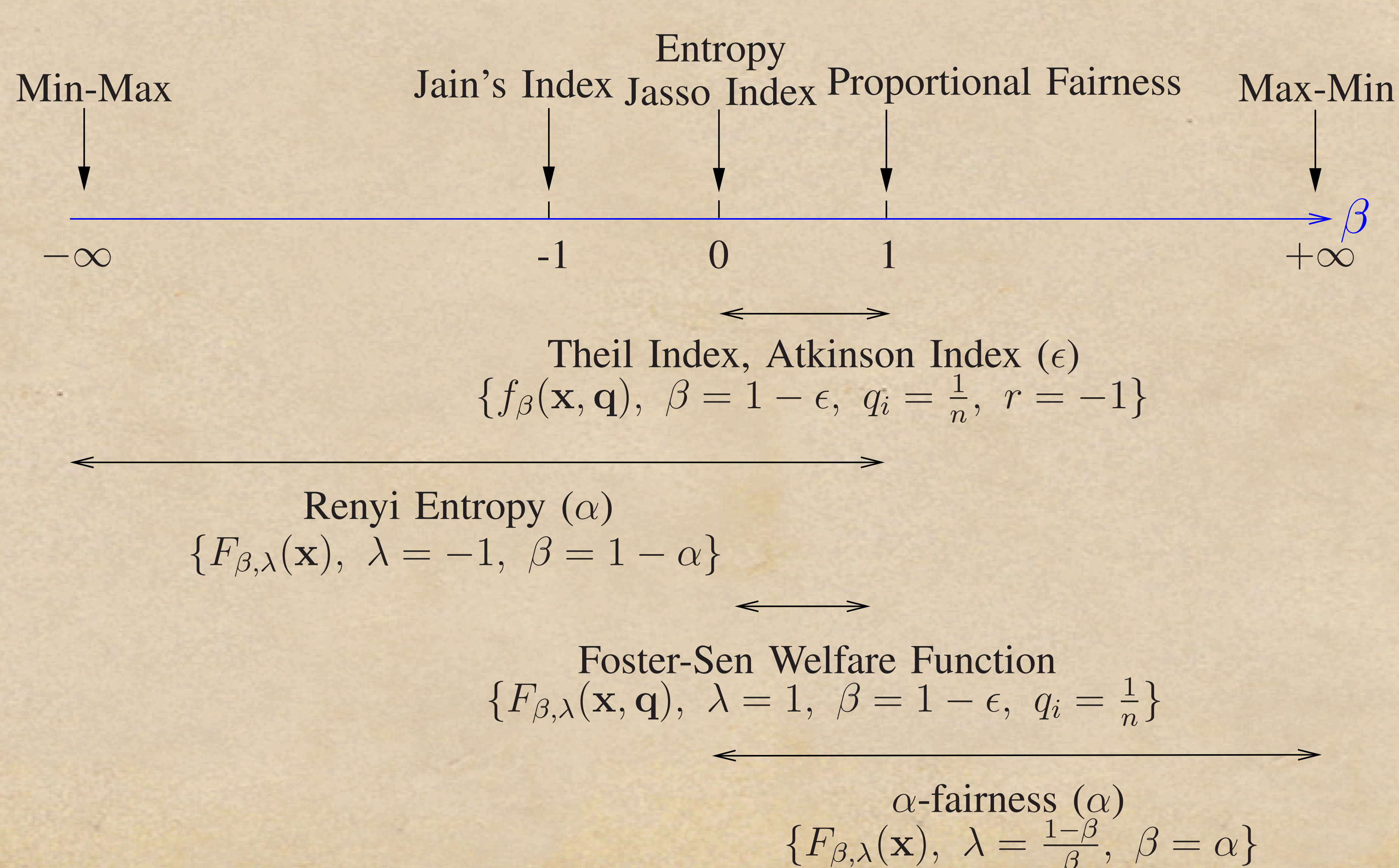
- Atkinson index as special case

$$A_\epsilon = 1 - \frac{1}{\text{mean}} \left(\frac{1}{n} \sum_i x_i^{1-\epsilon} \right)^{\frac{1}{1-\epsilon}}, \quad \epsilon \in [0, 1)$$

$$1 - \frac{1}{\text{mean}} \left(\prod_i x_i \right)^{\frac{1}{n}}, \quad \epsilon = 1$$

- Jasso, Theil indices
- Foster-Sen utility function

Unifying System-Wide Decomposable Fairness



Minimalistic Axioms?

- Partition
- Starvation
- Continuity and Saturation

Procedural Fairness

- Adams 1963. Equity: proportional to weights
- Voting paradoxes, Arrow's impossibility theorem
- Outcome vs. procedural fairness in legal systems
- Experiments and correlation analysis
- Thibaut and Walker 1975
- Leventhal's 6 rules 1980
- Fairness is as much about procedure as outcome

More Fun Than Expected

- Just a starter here (given a vector or two, look for a scaler valued function)

Thank You

www.princeton.edu/~chiangm