ELE539A: Optimization of Communication Systems
Lecture 10: Heterogeneous Congestion Control Protocols

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Lecture Outline

- Heterogeneous protocol
- Equilibrium
- Dynamics and control

Questions

Motivation 1: Coexistence of different congestion prices:

- Loss based TCP: Tahoe and Reno
- Delay based TCP: Vegas and FAST

Motivation 2: Per-user differential pricing

Can the number of equilibrium be:

0?

1?

2?

3?

∞?

Network Equilibrium

- L links: Fixed capacity c_l and has an intrinsic price p_l
- J different protocols: Type j sources react to effective prices p_l^j at links l in their paths:

$$p_l^j = m_l^j(p_l)$$

Network equilibrium condition:

- Flow constraint satisfied: $\mathbf{Rx} \preceq \mathbf{c}$
- Source rate maximizes local net utility (with effective path price): $x_s^j = \left(U_s^j\right)^{\prime -1}\left(q_s^j\right)$, where $\mathbf{q} = \mathbf{R}^T m(\mathbf{p})$
- Product of intrinsic price and rate mismatch is zero: $p_l(c_l - (\mathbf{Rx})_l) = 0$

No longer solves a social welfare maximization

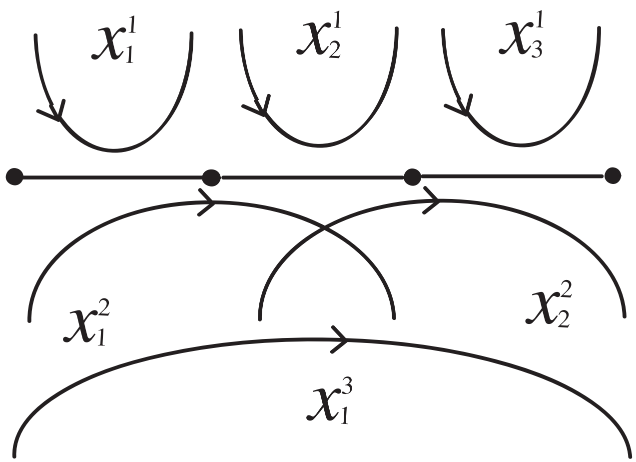
Existence of Equilibrium

Assumption 1: Price mapping functions $m_l^j(\cdot)$ are continuous and strictly increasing with $m_l^j(0) = 0$

Assumption 2: For large enough link price, source rate can be as small as desired

Theorem: There exists an equilibrium price $\mathbf{p}^*$ for any network $(\mathbf{R}, \mathbf{c}, \{U\}, \{m\})$

Example With Infinite Number of Equilibria



Concave quadratic utilities

All of the following are equilibrium prices

$$p_1 = p_3 = 1/8 + \epsilon$$

$$p_2 = 1/4 - 2\epsilon \quad \text{where} \quad \epsilon \in [0, 1/24]$$

Regular Networks

A network $(\mathbf{R}, \mathbf{c}, \{U\}, \{m\})$ is regular if all its equilibrium prices are locally unique

Theorem: For any routing $\mathbf{R}$, utilities $\{U\}$, and price mapping functions $\{m\}$, the set of link capacities $\mathbf{c}$ for which the network is not regular has measure zero in $\mathbf{R}^L$

Proof: use Sard's Theorem

Theorem: Number of equilibria for a regular network is finite

Implications: almost all networks have finite number of equilibria for heterogeneous congestion control protocols

A Further Fundamental Characterization

Define index $I(\mathbf{p})$ of an equilibrium price $\mathbf{p} \in E$ as 1 if det$(\mathbf{J}(\mathbf{p})) > 0$, and -1 if det$(\mathbf{J}(\mathbf{p})) < 0$

$$\mathbf{J}(\mathbf{p}) = \sum_j \mathbf{R}^j \text{diag}\left(\left(\frac{\partial^2 U_i^j}{\partial (x_i^j)^2}\right)^{-1}\right)(\mathbf{R}^j)^T \text{diag}\left(\frac{\partial m_l^j}{\partial p_l}\right)$$

Theorem:

$$\sum_{\mathbf{p} \in E} I(\mathbf{p}) = (-1)^L$$

Proof: use Poincare-Hopf Index Theorem from differential topology

Theorem: Number of equilibria is odd

Proof: both $I(\mathbf{p})$ and $(-1)^L$ are odd

Implications

An equilibrium is locally stable if $\mathbf{J}(\mathbf{p})$ is stable, *i.e.*, all eigenvalues have negative real part

Theorem: If all equilibria are locally stable, then there is a globally unique equilibrium

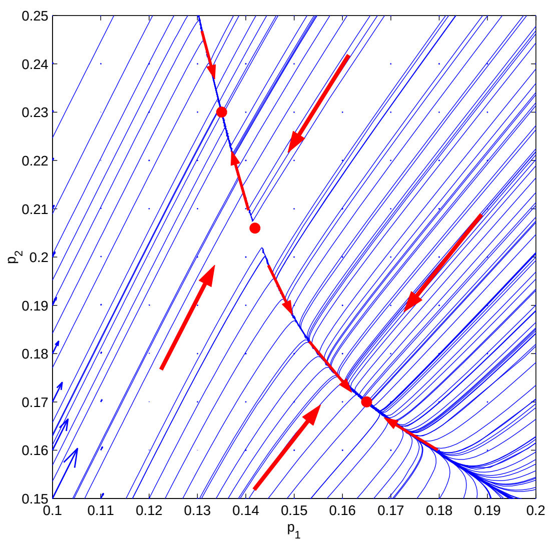
Proof: locally stable equilibrium has index $(-1)^L$

Implications: a local property of algorithms (local stability) implies a global property of a network (global uniqueness)

Example Revisited

Same topology and routing as in last Example, but using α utilities: $\frac{x^{1-\alpha}}{1-\alpha}$

Equilibria(p_1, p_2, p_3)	Eigenvalues	Index
$(0.135, 0.23, 0.135)$	$-0.21, -17.43, -26.73$	-1
$(0.1419, 0.2060, 0.1419)$	$0.21, -12.32, -22.40$	1
$(0.165, 0.17, 0.165)$	$-12.41, -1.67, -0.67$	-1

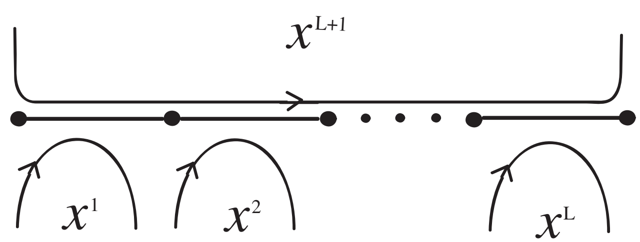


Special Cases for Global Uniqueness

Corollary: For linear and link-independent price mapping functions, there is a unique equilibrium

Corollary: For a network with at most two congested links, there is a unique equilibrium

Corollary: For a line network, there is a unique equilibrium



Homogeneity Implies Uniqueness

Theorem: Global uniqueness if one of the following is true:

For each $l = 1, \ldots, L$, $j = 1, \ldots, J$

$$\dot{m}_l^j \in \left[a_l, 2^{\frac{1}{L}} a_l\right] \qquad \text{for some } a_l > 0$$

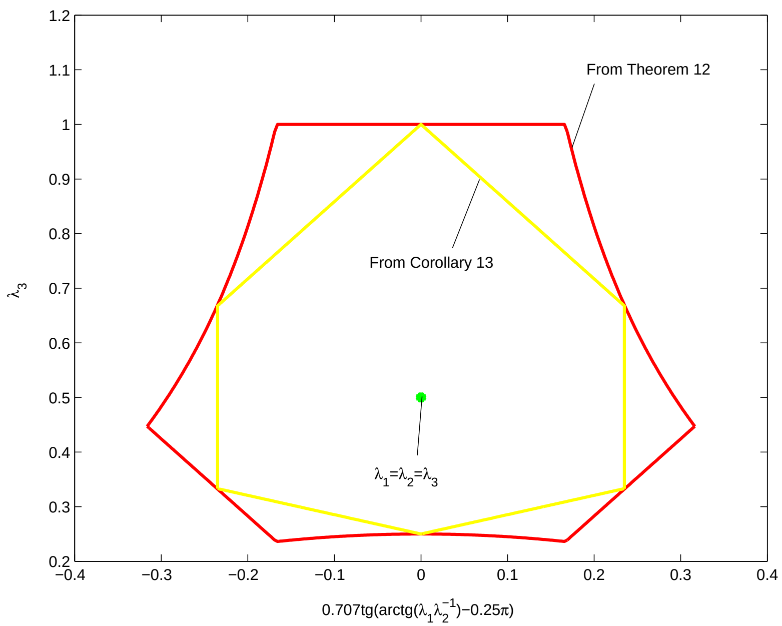
For all $j = 1, \ldots, J$, $l = 1, \ldots, L$

$$\dot{m}_l^j \in \left[a^j, 2^{\frac{1}{L}} a^j\right] \qquad \text{for some } a^j > 0$$

Homogeneity Implies Uniqueness

Practically important case: 2 protocols and 3 bottleneck links

Theorem: If $\dot{m}_l^1(p)/\dot{m}_l^2(p) \in [a, 2a]$ for some constant $a > 0$ for all l, there is a globally unique equilibrium



Conclusions

Can the number of equilibrium be:

0? No.

1? Maybe, can check with two different sufficient conditions

2? Almost never

3? Maybe

∞? Almost never

Experimental verifications: Tang, Wang, Hedge, Low 2005

Stability: Local stability whenever there is a global uniqueness

Design: Two (dual) method to design price mapping functions or end-user adaptations to steer the network to a desired equilibrium

Lecture Summary

- Vector field interpretation of heterogeneous protocols
- Bounded heterogeneity implies uniqueness and stability

Readings: A. Tang, J. Wang, S. H. Low, and M. Chiang, “Equilibrium of heterogeneous congestion control protocols”, *Proc. IEEE INFOCOM*, March 2005.