ELE539A: Optimization of Communication Systems
Lecture 10: Heterogeneous Congestion Control Protocols

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Lecture Outline

- Heterogeneous protocol
- Equilibrium
- Dynamics and control
Questions

Motivation 1: Coexistence of different congestion prices:
- Loss based TCP: Tahoe and Reno
- Delay based TCP: Vegas and FAST

Motivation 2: Per-user differential pricing

Can the number of equilibrium be:
0?
1?
2?
3?
∞?
Network Equilibrium

- $L$ links: Fixed capacity $c_l$ and has an intrinsic price $p_l$
- $J$ different protocols: Type $j$ sources react to effective prices $p^j_l$ at links $l$ in their paths:
  \[ p^j_l = m^j_l(p_l) \]

Network equilibrium condition:
- Flow constraint satisfied: $R x \preceq c$
- Source rate maximizes local net utility (with effective path price):
  \[ x^j_s = \left( U^j_s \right)^{-1} \left( q^j_s \right), \text{ where } q = R^T m(p) \]
- Product of intrinsic price and rate mismatch is zero:
  \[ p_l(c_l - (R x)_l) = 0 \]

No longer solves a social welfare maximization
Existence of Equilibrium

Assumption 1: Price mapping functions $m^j_l(\cdot)$ are continuous and strictly increasing with $m^j_l(0) = 0$

Assumption 2: For large enough link price, source rate can be as small as desired

Theorem: There exists an equilibrium price $p^*$ for any network $(R, c, \{U\}, \{m\})$
Example With Infinite Number of Equilibria

Concave quadratic utilities

All of the following are equilibrium prices

\[ p_1 = p_3 = \frac{1}{8} + \epsilon \]
\[ p_2 = \frac{1}{4} - 2\epsilon \quad \text{where} \quad \epsilon \in [0, 1/24] \]
Regular Networks

A network \((R, c, \{U\}, \{m\})\) is regular if all its equilibrium prices are locally unique

**Theorem:** For any routing \(R\), utilities \(\{U\}\), and price mapping functions \(\{m\}\), the set of link capacities \(c\) for which the network is not regular has measure zero in \(R^L\)

**Proof:** use *Sard's Theorem*

**Theorem:** Number of equilibria for a regular network is finite

**Implications:** almost all networks have finite number of equilibria for heterogeneous congestion control protocols
A Further Fundamental Characterization

Define *index* $I(p)$ of an equilibrium price $p \in E$ as 1 if $\det(J(p)) > 0$, and $-1$ if $\det(J(p)) < 0$

$$J(p) = \sum_j R^j \text{diag} \left( \left( \frac{\partial^2 U^j_i}{\partial (x^j_i)^2} \right)^{-1} \right) (R^j)^T \text{diag} \left( \frac{\partial m^j_i}{\partial p_l} \right)$$

Theorem:

$$\sum_{p \in E} I(p) = (-1)^L$$

Proof: use Poincare-Hopf Index Theorem from differential topology

Theorem: Number of equilibria is odd

Proof: both $I(p)$ and $(-1)^L$ are odd
Implications

An equilibrium is locally stable if $J(p)$ is stable, i.e., all eigenvalues have negative real part.

**Theorem:** If all equilibria are locally stable, then there is a globally unique equilibrium.

**Proof:** locally stable equilibrium has index $(-1)^L$.

**Implications:** a local property of algorithms (local stability) implies a global property of a network (global uniqueness).
Example Revisited

Same topology and routing as in last Example, but using $\alpha$ utilities:

$$\frac{x^{1-\alpha}}{1-\alpha}$$

<table>
<thead>
<tr>
<th>Equilibria($p_1, p_2, p_3$)</th>
<th>Eigenvalues</th>
<th>Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.135, 0.23, 0.135)</td>
<td>$-0.21, -17.43, -26.73$</td>
<td>-1</td>
</tr>
<tr>
<td>(0.1419, 0.2060, 0.1419)</td>
<td>$0.21, -12.32, -22.40$</td>
<td>1</td>
</tr>
<tr>
<td>(0.165, 0.17, 0.165)</td>
<td>$-12.41, -1.67, -0.67$</td>
<td>-1</td>
</tr>
</tbody>
</table>
**Special Cases for Global Uniqueness**

**Corollary:** For linear and link-independent price mapping functions, there is a unique equilibrium.

**Corollary:** For a network with at most two congested links, there is a unique equilibrium.

**Corollary:** For a line network, there is a unique equilibrium.
Homogeneity Implies Uniqueness

**Theorem:** Global uniqueness if one of the following is true:

For each $l = 1, \ldots, L$, $j = 1, \ldots, J$

$$\hat{m}^j_l \in [a_l, 2^{\frac{1}{L}} a_l]$$

for some $a_l > 0$

For all $j = 1, \ldots, J$, $l = 1, \ldots, L$

$$\hat{m}^j_l \in [a^j, 2^{\frac{1}{L}} a^j]$$

for some $a^j > 0$
**Homogeneity Implies Uniqueness**

Practically important case: 2 protocols and 3 bottleneck links

**Theorem:** If $\dot{m}_l^1(p)/\dot{m}_l^2(p) \in [a, 2a]$ for some constant $a > 0$ for all $l$, there is a globally unique equilibrium.
Conclusions

Can the number of equilibrium be:

0? No.

1? Maybe, can check with two different sufficient conditions

2? Almost never

3? Maybe

∞? Almost never

Experimental verifications: Tang, Wang, Hedge, Low 2005

Stability: Local stability whenever there is a global uniqueness

Design: Two (dual) method to design price mapping functions or end-user adaptations to steer the network to a desired equilibrium
Lecture Summary

- Vector field interpretation of heterogeneous protocols
- Bounded heterogeneity implies uniqueness and stability