**Outline**

- Case 1: Rate Reliability Tradeoff
- Case 2: TCP/IP Interactions

Many other cases in survey paper on “Layering As Optimization Decomposition”
Case 1

Rate Reliability Tradeoff
Reliability

Application needs at sources: utility of reliability

Physical layer possibilities on (some) links (e.g., DSL): adaptive coding and modulation

Intuition of the new opportunity:

- **Link tradeoff**: Fatter pipe, lower reliability
- **Source tradeoff**: Higher rate, lower quality

Signal quality and physical layer entirely missing from basic NUM
Problem Formulation

Assumptions: decode and reencode with small error probabilities

Reliability: $R_s$ for source $s$

Code rate: $r_{l,s}$ on link $l$ for source $s$

Error probability as a function of code rate: $E_l(r_{l,s})$

\[
\begin{align*}
\text{maximize} & \quad \sum_s U_s(x_s, R_s) \\
\text{subject to} & \quad R_s = 1 - \sum_{l \in L(s)} E_l(r_{l,s}), \quad \forall s \\
& \quad \sum_{s \in S(l)} \frac{x_s}{r_{l,s}} \leq C_l^{\text{max}}, \quad \forall l \\
& \quad x_s^{\text{min}} \leq x_s \leq x_s^{\text{max}}, \quad \forall s \\
& \quad 0 \leq r_{l,s} \leq 1, \quad \forall l, s \\
\text{variables} & \quad x, R, r
\end{align*}
\]
Overview

**Difficulty**: Neither convex nor separable problem

**Goal**: Derive globally optimal and distributed algorithm

- Develop such algorithms
- Extend pricing interpretation
- **Sufficient conditions** for convergence to global optimum
- **Techniques** to tackle nonconvexity and nonseparability issues
**Integrated Policy**

Each link maintains the same code rate for all sources traversing it

\[
\text{maximize} \quad \sum_{s} U_s(x_s, R_s) \\
\text{subject to} \quad R_s \leq 1 - \sum_{l \in L(s)} E_l(r_l), \quad \forall s \\
\sum_{s \in S(l)} \frac{x_s}{r_l} \leq C_{l}^{\text{max}}, \quad \forall l \\
\text{variables} \quad x, R, r
\]

Naturally decompose:

\[
\sum_{s \in S(l)} x_s \leq C_{l}^{\text{max}} r_l, \quad \forall l
\]
Approximation of $E_l(r_l)$:

\[
p_l \leq \exp(-NE_0(r_l))
\]

\[
E_0(r_l) = \max_{0 \leq \rho \leq 1} \max_Q [E_o(\rho, Q) - \rho r_l]
\]

\[
E_o(\rho, Q) = - \log \sum_{j=0}^{J-1} \left[ \sum_{k=0}^{K-1} Q(k)P(j|k)^{1/(1+\rho)} \right]^{1+\rho}
\]

Lemma:

If the absolute value of the first derivatives of $E_0(r_l)$ is bounded away from 0 and absolute value of the second derivative of $E_0(r_l)$ is upper bounded, then for a large enough codeword block length $N$, $E_l(r_l)$ is a convex function

Next: Use standard Lagrangian relaxation and distributed subgradient algorithm to develop distributed algorithm
Distributed Algorithm 1

Source problem and reliability price update at source $s$:

- **Source problem:**

  $$\text{maximize} \quad U_s(x_s, R_s) - \lambda^s(t)x_s - \mu_s(t)R_s$$

  $$\text{subject to} \quad x^\text{min}_s \leq x_s \leq x^\text{max}_s$$

  where $\lambda^s(t) = \sum_{l \in L(s)} \lambda_l(t)$ is the end-to-end congestion price at iteration $t$

- **Reliability price update:**

  $$\mu_s(t + 1) = [\mu_s(t) - \alpha(t)(R^s(t) - R_s(t))]^+$$

  where $R^s(t) = 1 - \sum_{l \in L(s)} E_l(r_l(t))$ is the end-to-end reliability at iteration $t$
Link problem and congestion price update at link $l$:

- **Link problem:**
  
  \[
  \text{maximize} \quad \lambda_l(t) r_l C_l^{max} - \mu_l(t) E_l(r_l) \\
  \text{subject to} \quad 0 \leq r_l \leq 1
  \]

  where $\mu_l(t) = \sum_{s \in S(l)} \mu_s(t)$ is the aggregate reliability price paid by sources using link $l$ at iteration $t$

- **Congestion price update:**
  
  \[
  \lambda_l(t + 1) = \left[ \lambda_l(t) - \alpha(t) \left( r_l(t) C_l^{max} - x^l(t) \right) \right]^+
  \]

  where $x^l(t) = \sum_{s \in S(l)} x_s(t)$ is the aggregate information rate on link $l$ at iteration $t$
**Pricing Interpretation**

- **Source problem**: maximize total net utility on rate (with total congestion price) and **reliability** (with signal quality price)
- **Source algorithm**: local solution of source problem (2 variables) **updates signal quality price**

- **Network problem**: maximize net revenue:
  receive revenue from rate
  pay price for unreliability
- **Link algorithm**: update link congestion price
Theorem: Distributed Algorithm 1 converges to the **globally optimal rate-reliability tradeoff** for sufficiently strong codes.
Differentiated Policy

Each link may give a different code rate for each of the sources traversing it

- Per-flow state needed
- Better rate-reliability tradeoff

\[
\begin{align*}
\text{maximize} & \quad \sum_s U_s(x_s, R_s) \\
\text{subject to} & \quad R_s \leq 1 - \sum_{l \in L(s)} E_l(r_{l,s}), \quad \forall s \\
& \quad \sum_{s \in S(l)} \frac{x_s}{r_{l,s}} \leq C_{l}^{\text{max}}, \quad \forall l \\
\text{variables} & \quad x, R, r
\end{align*}
\]
Difficulty 2: Coupling

Step 1: Introduce auxiliary variables:

\[
\begin{align*}
\text{maximize} \quad & \sum_s U_s(x_s, R_s) \\
\text{subject to} \quad & R_s \leq 1 - \sum_{l \in L(s)} E_l(r_{l,s}), \ \forall s \\
& \frac{x_s}{r_{l,s}} \leq c_{l,s}, \ \forall l, s \in S(l) \\
& \sum_{s \in S(l)} c_{l,s} \leq C_{l}^{\text{max}}, \ \forall l
\end{align*}
\]

Step 2: Log change of variables:

\[
\begin{align*}
\text{maximize} \quad & \sum_s U'_s(x'_s, R_s) \\
\text{subject to} \quad & R_s \leq 1 - \sum_{l \in L(s)} E_l(r_{l,s}), \ \forall s \\
& x'_s - \log r_{l,s} \leq \log c_{l,s}, \ \forall l, s \in S(l) \\
& \sum_{s \in S(l)} c_{l,s} \leq C_{l}^{\text{max}}, \ \forall l
\end{align*}
\]

Separable problem but \( U'_s(x'_s, R_s) \) may not be concave
Difficult 2: Coupling

Step 3: Concavity condition

\[ g_s(x_s, R_s) = \frac{\partial^2 U_s(x_s, R_s)}{\partial x_s^2} x_s + \frac{\partial U_s(x_s)}{\partial x_s}, \]

\[ h_s(x_s, R_s) = \left( \left( \frac{\partial^2 U_s(x_s, R_s)}{\partial x_s \partial R_s} \right)^2 \right), \]

\[ + \frac{\partial^2 U_s(x_s, R_s)}{\partial x_s^2} \frac{\partial^2 U_s(x_s, R_s)}{\partial R_s^2} x_s \]

\[ - \frac{\partial^2 U_s(x_s, R_s)}{\partial R_s^2} \frac{\partial U_s(x_s, R_s)}{\partial x_s}, \]

and

\[ q_s(x_s, R_s) = \frac{\partial^2 U_s(x_s, R_s)}{\partial R_s^2}. \]

**Lemma:** If \( g_s(x_s, R_s) < 0, h_s(x_s, R_s) < 0, \) and \( q_s(x_s, R_s) < 0, \) then \( U'_s(x'_s, R_s) \) is a concave function of \( x'_s \) and \( R_s \).
Difficulty 2: Coupling

Special case 1: \(\alpha\)-fair utilities

\[
U_s(x_s, R_s) = \begin{cases} 
\log x_s R_s, & \text{if } \alpha = 1 \\
(1 - \alpha)^{-1}(x_s R_s)^{1-\alpha}, & \text{otherwise}
\end{cases}
\]

If \(\alpha \geq 1\), conditions for concavity is satisfied

Special case 2: if \(U_s\) is additive in \(x_s\) and \(R_s\), its curvature needs to be not just negative but bounded away from 0 by as much as \(-\frac{dU_x^x(x_s)}{x_s dx_s}\), i.e., the application represented by this utility function must be elastic enough.
Distributed Algorithm 2

All descriptions same as in Algorithm 1 except one:

- Link problems:
  
  **Bandwidth allocation problem**

  maximize \( \sum_{s \in S(l)} \lambda_{l,s}(t) \log c_{l,s} \)

  subject to \( \sum_{s \in S(l)} c_{l,s} \leq C_{l}^{max} \)

- **Code rate allocation problem** for source \( s, s \in S(l) \)

  maximize \( \lambda_{l,s}(t) \log r_{l,s} - \mu_{s}(t)E_{l}(r_{l,s}) \)

  subject to \( 0 \leq r_{l,s} \leq 1 \)
Numerical Example

\[
U_s(x_s, R_s) = a_s \frac{x_s^{1-\alpha} - x_s^{\min(1-\alpha)}}{x_s^{\max(1-\alpha)} - x_s^{\min(1-\alpha)}} + (1 - a_s) \frac{R_s^{(1-\alpha)} - R_s^{\min(1-\alpha)}}{R_s^{\max(1-\alpha)} - R_s^{\min(1-\alpha)}}
\]

Case 1: \(a_s = a\)

Case 2: \(a_s = \begin{cases} 
0.5 - v, & \text{if } s \text{ is an odd number} \\
0.5 + v, & \text{if } s \text{ is an even number}
\end{cases}\)
Numerical Example
Numerical Example

![Graphs showing data rate (Mbps) vs. Reliability, and Network utility vs. \( v \)]
Case 2

TCP/IP Interactions
TCP/IP Interaction

Assumptions:
- TCP: dual-based congestion control
- IP: single, dynamic, shortest path routing
- TCP timescale much shorter than IP timescale

Goals:
- Utility attained at equilibrium
- Stability of interactions
- Implications to routing and link capacity provisioning
**Notation**

Path topology constant: $K^i$ acyclic paths for source $i$ represented by a $L \times K^i$ 0-1 matrix $H^i$:

$$H^i_{lj} = \begin{cases} 1, & \text{if path } j \text{ of source } i \text{ uses link } l \\ 0, & \text{otherwise.} \end{cases}$$

$\mathcal{H}^i$: set of all columns of $H^i$ (all the available paths to $i$ under single-path routing)

$L \times K = \sum_i K^i$ overall topology matrix: $H = [H^1 \ldots H^N]$

Path selection variable: $w^i$: $K^i \times 1$ vector where the $j$th entry represents the fraction of $i$'s flow on its $j$th path. $w^i_j \geq 0 \ \forall j$, and $1^T w^i = 1$

Collect vectors $w^i$, $i = 1, \ldots, N$, into $K \times N$ block-diagonal matrix $W$

Set of single path routing:

$\mathcal{W}_s = \{W | W = \text{diag}(w^1, \ldots, w^N) \in \{0, 1\}^{K \times N}, 1^T w^i = 1 \}$

Set of multipath routing:
\[ \mathcal{W}_m = \{ W | W = \text{diag}(w^1, \ldots, w^N) \in [0, 1]^{K \times N}, 1^T w^i = 1 \} \]

**Routing:** \( L \times N \) routing matrix \( R = HW \)

Set of all single-path routing matrices: \( \mathcal{R}_s = \{ R \mid R = HW, W \in \mathcal{W}_s \} \)

Set of all multipath routing matrices: \( \mathcal{R}_m = \{ R \mid R = HW, W \in \mathcal{W}_m \} \)

Single-path routing matrix in \( \mathcal{R}_s \) is an 0-1 matrix:

\[
R_{li} = \begin{cases} 
1, & \text{if link } l \text{ is in a path of source } i \\
0, & \text{otherwise.}
\end{cases}
\]

Multipath routing matrix in \( \mathcal{R}_m \):

\[
R_{li} \begin{cases} 
> 0, & \text{if link } l \text{ is in a path of source } i \\
= 0, & \text{otherwise.}
\end{cases}
\]

Path of source \( i \): \( r^i = [R_{1i} \ldots R_{Li}]^T \), the \( i \)th column of \( R \)
TCP/IP Equilibrium

Link cost at time $t$: $d_l(t) = a p_l(t) + b \tau_l$

Static component: $\tau_l$. Dynamic component: $p_l(t)$

TCP/IP equilibrium $R^*, x^*, p^*$ satisfies

$$r^i(t + 1) = \arg \min_{r^i \in \mathcal{H}^i} \sum_l (a p_l(t) + b \tau_l) r^i_l, \text{ for all } i$$

$$\sum_l R_{li}(t) p_l(t) = U'_i(x_i(t)) \text{ for all } i$$

$$\sum_i R_{li}(t) x_i(t) \begin{cases} \leq c_l & \text{if } p_l(t) \geq 0 \\ = c_l & \text{if } p_l(t) > 0 \end{cases} \text{ for all } l$$

$$x(t), p(t) \geq 0$$
Properties of Equilibrium

Case 1: $a = 0$ and $b > 0$ Purely static routing
- Trivial: Equilibrium always exist
- Open: May not solve any joint optimization problem

Case 2: $a > 0$ and $b = 0$ Purely dynamic routing
- Joint NUM solved by TCP/IP equilibrium iff no duality gap
- No ‘cost of not splitting’ iff no duality gap
- Joint NUM is NP-hard

Case 3: $a > 0$ and $b > 0$ General case
- Open: Not even sure about existence of equilibrium
Joint NUM Problem

Primal problem of NUM over $R$ and $x$:

\[
\max_{R \in \mathcal{R}_s} \max_{x \geq 0} \sum_i U_i(x_i) \quad \text{s. t.} \quad Rx \leq c
\]

Dual problem

\[
\min_{p \geq 0} \sum_i \max_{x_i \geq 0} \left( U_i(x_i) - x_i \min_{r^i \in \mathcal{H}^i} \sum_l R_{li}p_l \right) + \sum_l c_l p_l
\]

$r^i$ is the $i$th column of $R$ with $r^i_l = R_{li}$

Is TCP/IP (with purely dynamic routing) solving the above problems?

Yes, if TCP/IP equilibrium exists
Characterization

**Theorem:** TCP/IP equilibrium exists iff no duality gap, in which case it solves joint NUM

\[
V_{sp} = \max_{R \in \mathcal{R}_s, x \geq 0} \min_{p \geq 0} L(R, x, p)
\]

\[
V_{sd} = \min_{p \geq 0} \max_{R \in \mathcal{R}_s, x \geq 0} L(R, x, p)
\]

\[
V_{mp} = \max_{R \in \mathcal{R}_m, x \geq 0} \min_{p \geq 0} L(R, x, p)
\]

\[
V_{md} = \min_{p \geq 0} \max_{R \in \mathcal{R}_m, x \geq 0} L(R, x, p)
\]

**Theorem:** \( V_{sp} \leq V_{sd} = V_{md} = V_{mp} \)

Duality gap comes from single-path integer constraint
Equilibrium always exists. But there is a tradeoff between

- Network utility achieved
- Stability of routing
Utility Stability Tradeoff

Continuous ring network model: routing is represented by a scalar.

Theorem: Purely dynamic routing maximizes utility but routing oscillates between 0 and $N$ (except when starting from equilibrium).

Needs to set $b = 1$ and investigate effect of $a$ on stability.

Theorem: For sufficiently small $a$, routing is stable. For sufficiently large $a$, routing oscillates. Utility attained converges to optimal utility as $a \to \infty$. 
Summary

- Layering As Optimization Decomposition: Analysis and Design (one example for each direction)

- Nonconvexity and nonseparability issues

- Congestion price (queuing delay) often the right “layering price”, but not always

