On the Asymptotic Behavior of Selfish Transmitters Sharing a Common Channel

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***Abstract*—In a multiple-access communication network, nodes must compete for scarce communication resources such as bandwidth. This paper analyzes the asymptotic behavior of a multiple-access network comprising a large number of *selfish* transmitters competing for access to a common wireless communication channel, and having different utility functions for determining their strategies. A necessary and sufficient condition is given for the total number of packet arrivals from selfish transmitters to converge in distribution. The asymptotic packet arrival distribution at Nash equilibrium is shown to be a mixture of a Poisson distribution and finitely many Bernoulli distributions.**

I. Introduction

A fundamental problem arising in communications is the contention of transmitters for access to a common wireless communication channel in order to communicate with their intended receivers. For example, mobile users compete with one another for uplink reservations in currently existing cellular systems. Furthermore, due to the broadcast nature of the wireless medium, such situations are commonplace in many types of wireless networks. As a result, this problem has been received considerable attention in the literature, and issues such as packet arrival rates required to stabilize transmitters' queues, channel throughput, etc., are relatively well-understood when nodes obey to predetermined rules for choosing their transmission probabilities and back-off strategies (see [1], [2] and [3]).

On the other hand, understanding the behavior of *selfish* transmitters sharing a common wireless channel is a relatively recent problem. One way of dealing with such selfish behavior in communication networks is to push all the decision-making burden to individual nodes. Nodes selfishly decide what to do by sensing their local environments with the aim of maximizing their own utilities. Such an approach by its very nature results in distributed and scalable network control and management. *Game theory* provides the necessary mathematical tools to analyze the behavior of networks under such conditions. Examples of such approaches are found in [4] and [5], in which the authors use a game-theoretic approach to understand the behavior of selfish transmitters when they share a common wireless communication channel via the slotted ALOHA protocol.

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Game theory has an important role to play in the design of large networks in general, and that of wireless networks in particular (see, e.g. [6] and [7]). Game theory and the related field of mechanism design bring key tools into play - equilibrium concepts, utility functions, and signaling/bargaining - that are critical to the understanding of communication networks' dynamics and to the design of distributed network control algorithms for them. For example, in [8], authors reverse engineer exponential type back-off protocols for ad-hoc networks by using game theoretic tools, and prove that such protocols implicitly participate in a non-cooperative game trying to maximize selfish local utilities of network nodes. This result further reveals that game theory and the related field of mechanism design are not only important to understand the behavior of communication networks with selfish agents but also vital for understanding the intricate properties of existing medium access control (MAC) and contention resolution protocols for communication networks.

In this paper, we focus on the asymptotic properties of multiple-access communication networks in which selfish nodes share a common wireless communication channel to communicate with their intended receivers. We consider a heterogeneous multiple-access communication network in which selfish transmitters are allowed to have different utility functions from one another. We first give a complete characterization for all Nash equilibria of such a multiple-access communication network. Then, by building on this result, we obtain a necessary and sufficient condition for the total number of packet arrivals from selfish transmitters to converge in distribution as the number of selfish transmitters contending for channel access increases. We also identify the asymptotic form of the packet arrival distribution. In particular, we show that asymptotic packet arrival distribution at Nash equilibrium is a mixture of a Poisson distribution and finitely many Bernoulli distributions.

An important practical implication of this result for mathematical modeling of multiple-access communication networks is that packet arrivals to a common wireless communication cannot be assumed to be a pure Poisson process in multiple-access communication networks consisting of infinitely many selfish transmitters having different utility functions. Rather, asymptotic packet arrival distribution consists of two parts, one of which is a pure Poisson distribution, and the other one is a convolution of finitely many Bernoulli distributions.

A. Related Work

Two related studies are described in [4] and [5]. In these studies, the authors model the behavior of slotted ALOHA with selfish transmitters by using repeated games, and they thereby analyze the performance and stability properties of slotted ALOHA in this situation. However, they consider only the homogenous case in which selfish nodes have identical utility functions, and prove that there exists a symmetric Nash equilibrium. In another study [9], the authors give a similar game theoretic analysis of slotted ALOHA. They also assume that all the nodes are identical and consider only the symmetric Nash equilibrium. Furthermore, [10] and [11] consider the asymptotic channel throughput and the asymptotic packet arrival distribution in the homogenous case for the same problem set-up. In particular, the asymptotic form of the channel throughput and the packet arrival distribution as well as bounds on convergence are obtained there for the homogenous case.

Unlike the existing work in the literature, this paper will concentrate on the asymptotic packet arrival distribution in the more general and realistic situation in which selfish transmitters having different utility functions contend for access to a common wireless communication channel. We provide a necessary and sufficient condition for the convergence in distribution of the total number of packet arrivals as the number of selfish transmitters increases without bound. We also specify the form of the asymptotic packet arrival distribution.

B. A Note on Notation

We assume a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which all random variables are defined. Convergence *with probability one* (*w.p.1*) is taken with respect to the probability measure $\mathbb{P}$. $Po(\lambda)$ and $Bern(p)$ indicate the Poisson distribution with mean λ and the 0-1 Bernoulli distribution with mean p, respectively, as well as generic random variables with these distributions.

$\mathbb{N}$ and $\mathbb{Z}$ represent the set of natural numbers and the set of integers, respectively. $\mathbb{R}_+$ and $\mathbb{R}_+^n$ represent the set of positive real numbers and the nth-fold cartesian product of the set of positive real numbers for $n \geq 2$, respectively. For a given set $\mathcal{N}_0$, $|\mathcal{N}_0|$ will represent the cardinality of the set. We say a real valued function $f(t)$ is $O(t)$ if there exists a constant $c > 0$ such that $|f(t)| \leq c|t|$ as t goes to zero.

For any given discrete distributions μ and ν on $\mathbb{Z}$, $d_V(\mu, \nu)$ denotes the *variational distance* between them:

$$d_V(\mu, \nu) = \sum_{z \in \mathbb{Z}} |\mu(z) - \nu(z)|.$$

If X and Y are random variables with distributions μ and ν, we sometimes write $d_V(X, Y)$ in stead of $d_V(\mu, \nu)$ for ease of understanding. If one of the arguments of d_V contains a summation of random variables, this refers to the convolution of their respective distributions. If a sequence of probability distributions $\{\mu_n\}_{n=1}^{\infty}$ converges (in the usual sense of convergence in distribution) to another probability distribution μ, we represent this convergence by $\mu_n \Rightarrow \mu$ as $n \to \infty$. $Var(X)$ denotes the variance of the random variable X.

II. Problem Setup and the Nash Equilibria of the One-shot Random Access Game

To investigate the behavior of a multiple-access communication network consisting of large number of selfish transmitters, we consider the network model depicted in Fig. 1. In Fig. 1, each transmitter in the transmitter set has an intended receiver in the receiver set. In the context of cellular networks, the transmitter set consists of mobile users requesting uplink reservations to communicate with a base station. In a more general setting, it can be thought of as containing some number of wireless transmitters that are closely located in a wireless ad-hoc network, and that are willing to communicate with another close-by node. The results in this paper can be viewed as characterizing the local behavior of dense wireless networks containing selfish nodes and using a collision channel model at the medium access control layer. The collision channel model has been extensively used in the past (e.g., [12], [13]), and it is appropriate to characterize the behavior of networks using no power control and containing nodes with single packet detection capabilities. The protocol model defined in [14] is a variation of the collision model.

A. Game Definition

We assume that transmitter nodes always have packets to transmit, and a transmission fails if there are more than one transmissions at the same time. The cost of unsuccessful transmission of node i is $c_i \in (0, \infty)$. If a transmission is successful, the node that transmitted its packet successfully gets utility 1 unit. We model this situation by using a strategic game $G(n, \mathbf{c})$, which is defined formally as follows.

Definition 1: A heterogenous one-shot random access game with n transmitter nodes is the game $G(n, \mathbf{c}) = \langle \mathcal{N}, (\mathcal{A}_i)_{i \in \mathcal{N}}, (u_i)_{i \in \mathcal{N}} \rangle$ such that $\mathcal{N} = \{1, 2, \ldots, n\}$ is the set of transmitters, $\mathcal{A}_i = \{0, 1\}$ for all $i \in \mathcal{N}$, where $\mathcal{A}_i$ is the set of actions for node i and 1 means transmission and 0 means back-off, $\mathbf{c} = (c_i)_{i \in \mathcal{N}}$ where c_i is the cost of unsuccessful transmission for node i, and the utility function u_i for each $i \in \mathcal{N}$ is defined as

$$\begin{aligned} u_i(\mathbf{a}) &= 0 && \textit{if } a_i = 0, \\ u_i(\mathbf{a}) &= 1 && \textit{if } a_i = 1 \textit{ and } a_j = 0, \forall j \neq i, \\ u_i(\mathbf{a}) &= -c_i && \textit{if } a_i = 1 \textit{ and } \exists j \neq i \textit{ such that } a_j = 1. \end{aligned}$$

If $c_i = c > 0$ for all $i \in \mathcal{N}$, then we will denote $G(n, \mathbf{c})$ by $G(n, c)$, and call it a *homogenous* one-shot random access game. c_i is node i's subjective evaluation for failed packet transmissions. Every failed packet results in an increase of the delay of a packet until it is delivered to its intended destination, and the amount of energy spent per packet until successful delivery. Therefore, it is very natural to expect that selfish nodes evaluate failed packet transmissions differently depending on their battery energy limitations and how delay tolerant they are. For example, it is anticipated that failed transmissions incur high costs for nodes having very limited battery energy.

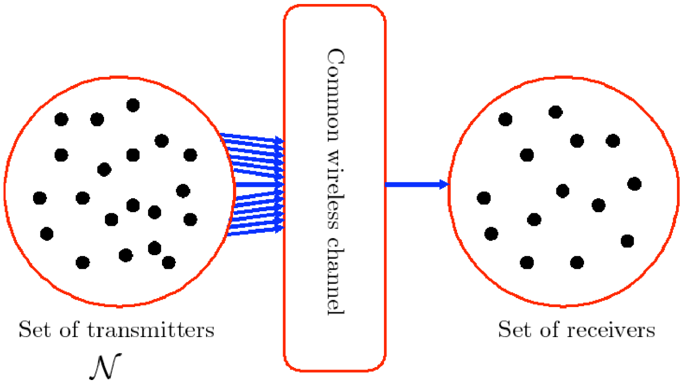

Fig. 1. Network model in which n selfish transmitters contend for the access of a common wireless communication channel to communicate with their intended receivers in the receiver set.

B. Nash Equilibria of $G(n, \mathbf{c})$

Recall that a Nash equilibrium is an equilibrium point at which none of the transmitters has an incentive to deviate. Therefore, the commonly used transmission probability vector at which all nodes back-off with probability one is not a Nash equilibrium of $G(n, \mathbf{c})$ since any node can obtain positive utility by setting its transmission probability to a positive number given the fact that others do not transmit. Thus, there is an incentive for nodes to deviate from the strategy profile at which all of them back-off. As a result, at Nash equilibria of $G(n, \mathbf{c})$, we expect to observe some of the transmitters transmitting with positive probabilities and the rest backing-off with probability one.

To further investigate this point, let $\pi : \bigcup_{n=2}^{\infty} \mathbb{R}_+^n \to \mathbb{R}_+$ be such that for any $\mathbf{c} \in \bigcup_{n=2}^{\infty} \mathbb{R}_+^n$, $\pi(\mathbf{c}) = \prod_i \frac{c_i}{1+c_i}$. The following theorem from [11] characterizes the Nash equilibria of this game.

Theorem 1: Let $X_i^{(n)}$ denote the action chosen by transmitter $i \in \mathcal{N}$, $\mathbf{c} \in \mathbb{R}^n$ and $\mathcal{N}_0 \subseteq \mathcal{N}$ with $2 \leq |\mathcal{N}_0| \leq n$. Then, $G(n, \mathbf{c})$ has n pure-strategy Nash equilibria. Moreover, any mixed-strategy profile such that nodes in $\mathcal{N}_0$ transmit with some positive probability, and nodes in $\mathcal{N} - \mathcal{N}_0$ back-off with probability 1 is a Nash equilibrium if and only if

$$\mathbb{P}\{X_i^{(n)} = 1\} = 1 - \left(\frac{1+c_i}{c_i}\right) (\pi(\mathbf{c}'))^{\frac{1}{|\mathcal{N}_0|-1}}$$

for $i \in \mathcal{N}_0$, and $\frac{c_i}{1+c_i} \overset{(\geq)}{>} \pi(\mathbf{c}')^{\frac{1}{|\mathcal{N}_0|-1}}$ for all $i \in \mathcal{N}$ (with $\geq$ if $i \in \mathcal{N} - \mathcal{N}_0$), where $\mathbf{c}' = (c_i)_{i \in \mathcal{N}_0}$.

Note that $X_i^{(n)}$ in Theorem 1 is a 0-1 random variable showing the action chosen by transmitter $i \in \mathcal{N}$ when the game $G(n, \mathbf{c})$ is played. 0 means back-off, and 1 means transmit. Therefore, if S_n represents the total number of packet arrivals when there are n transmitters contending for channel access, S_n is then equal to

$$S_n = \sum_{i=1}^{n} X_i^{(n)}.$$

We represent the transmission probability of transmitter i by $p_{i,n}$ when there are n transmitters contending for channel access. We call a Nash equilibrium a *fully-mixed Nash equilibrium* (FMNE) if all transmitters transmit with some positive probability at this equilibrium (i.e., $p_{i,n} > 0$ for all $i \in \mathcal{N}$).

C. Review: Homogenous Case

We briefly mention the form of the asymptotic distribution of the total number of packet arrivals when all transmitters have identical utility functions. In this case, the necessary and sufficient condition given in Theorem 1 can be satisfied for any subset $\mathcal{N}_0$ of $\mathcal{N}$ other than the empty set for proper choice of the nodes' transmission probabilities. Therefore, for any given $\mathcal{N}_0 \subseteq \mathcal{N}$ (apart from the empty set), a Nash equilibrium at which only the transmitters in $\mathcal{N}_0$ transmit with some positive probability, and the rest of them back-off with probability one exists. At such a Nash equilibrium, the transmission probabilities of transmitters in $\mathcal{N}_0$ are all equal to

$$p = 1 - \left(\frac{c}{1+c}\right)^{\frac{1}{|\mathcal{N}_0|-1}}.$$

Thus, transmitters transmit with probability $p = 1 - \left(\frac{c}{1+c}\right)^{\frac{1}{n-1}}$ at the FMNE. Hence, at the FMNE of the homogenous random access game, S_n becomes a binomial random variable with success probability $p = 1 - \left(\frac{c}{1+c}\right)^{\frac{1}{n-1}}$. Since $n \cdot \left(1 - \left(\frac{c}{1+c}\right)^{\frac{1}{n-1}}\right)$ approaches $-\log\left(\frac{c}{1+c}\right)$ as n goes to infinity, S_n converges, in distribution, to a Poisson distribution with mean $-\log\left(\frac{c}{1+c}\right)$, which can be shown by using the Poisson approximation to the binomial distribution [16]. Further details can be found in [10]. For the rest of the paper, our aim is to prove a similar limit theorem for S_n in the more general case when nodes do not have identical utility functions. We first give a counter example showing that the limiting distribution of S_n cannot always be a pure Poisson distribution when nodes have different utility functions. In this latter case, we then, however, show that it can be arbitrarily closely approximated in distribution by a summation of finitely many independent Bernoulli random variables and a Poisson random variable.

III. Limiting Behavior of S_n in the Heterogeneous Case

We start our discussion with an example illustrating that the Poisson type convergence does not occur in general in the heterogeneous utility case. This result, while somewhat negative, will shed light on the general form of the limiting distributions for S_n. In this example, the limiting distribution of the packet arrivals will be a mixture of a Poisson distribution and several Bernoulli distributions.

A. Example

We consider the FMNE of the one-shot random access game, and let

$$\mathbf{c}_n = \big(M_1, M_2, \ldots, M_l, \underbrace{1, 1, \ldots, 1}_{n-l \text{ of them}}\big).$$

By Theorem 1, $G(n, \mathbf{c}_n)$ has an FMNE if and only if the following conditions are satisfied:

$$\frac{M_i}{1+M_i} > \left(\frac{1}{2}\right)^{\frac{n-l}{n-1}} \prod_{j=1}^{l} \left(\frac{M_j}{1+M_j}\right)^{\frac{1}{n-1}} \quad \text{for } 1 \leq i \leq l, \quad (1)$$

and

$$\left(\frac{1}{2}\right)^{l-1} > \prod_{j=1}^{l} \frac{M_j}{1+M_j} \quad \text{for } l+1 \leq i \leq n. \quad (2)$$

Since the right-hand side of (1) approaches $\frac{1}{2}$ as n goes to infinity, we must choose $M_i > 1$ for all $i \in \{1, 2, ..., l\}$ to have the FMNE for all sufficiently large n. Any choice of $M_1, M_2, \ldots, M_l$ such that $M_i > 1$ for all $i \in \{1, 2, ..., l\}$ and

$$\prod_{j=1}^{l} \frac{M_j}{1+M_j} < \left(\frac{1}{2}\right)^{l-1}$$

is good for our purposes. One way of choosing such M_i's is to make all of the $\frac{M_i}{1+M_i}$'s smaller than $\left(\frac{1}{2}\right)^{\frac{l-1}{l}}$, which corresponds to

$$M_1, M_2, \ldots, M_l \in \left(1, \frac{1}{2^{\frac{l-1}{l}} - 1}\right).$$

For appropriately chosen M_i, $1 \leq i \leq l$, we have the following transmission probabilities:

$$p_{i,n} = 1 - \frac{1+M_i}{M_i} \left(\frac{1}{2}\right)^{\frac{n-l}{n-1}} \prod_{j=1}^{l} \left(\frac{M_j}{1+M_j}\right)^{\frac{1}{n-1}} \quad \text{for } 1 \leq i \leq l, \quad (3)$$

and

$$p_{i,n} = 1 - 2\left(\frac{1}{2}\right)^{\frac{n-l}{n-1}} \prod_{j=1}^{l} \left(\frac{M_j}{1+M_j}\right)^{\frac{1}{n-1}} \quad \text{for } l+1 \leq i \leq n. \quad (4)$$

Define $Y_n = \sum_{i=l+1}^{n} X_i^{(n)}$. Then,

$$S_n = \sum_{i=1}^{l} X_i^{(n)} + Y_n.$$

Observe that $p_{\max}^{(n)} \triangleq \max_{l+1 \leq i \leq n} p_{i,n} \to 0$ and

$$\sum_{i=l+1}^{n} p_{i,n} \to \log\left(2^{1-l}\right) + \sum_{j=1}^{l} \log\left(1 + \frac{1}{M_j}\right) \quad (5)$$

as $n \to \infty$. Therefore,

$$Y_n \Rightarrow Po\left(\log\left(2^{1-l}\right) + \sum_{j=1}^{l} \log\left(1 + \frac{1}{M_j}\right)\right), \quad (6)$$

and

$$X_i^{(n)} \Rightarrow Bern\left(1 - \frac{1+M_i}{2M_i}\right) \quad \text{for } 1 \leq i \leq l. \quad (7)$$

As a result, we conclude, by using the continuity theorem and the independence of the random variables Y_n and $X_i^{(n)}$, that

$$S_n \Rightarrow Po\left(\log\left(2^{1-l}\right) + \sum_{i=1}^{l} \log\left(1 + \frac{1}{M_i}\right)\right) + \sum_{i=1}^{l} Bern\left(1 - \frac{1+M_i}{2M_i}\right). \quad (8)$$

One interesting feature of this example is that we cannot find infinitely many M_i's that are uniformly bounded away from 1, since $\frac{1}{2^{\frac{l-1}{l}} - 1} \to 1$ as $l \to \infty$. As l increases, we must choose $M_1, M_2, \ldots, M_l$ closer to 1 in order to satisfy the necessary and sufficient condition for the existence of the FMNE. For example, if $l = 10$, then $\frac{1}{2^{\frac{l-1}{l}} - 1} \cong 1.155$, whereas $\frac{1}{2^{\frac{l-1}{l}} - 1} \cong 2.414$ when $l = 2$. It is possible to make some of the M_i's bigger but in this case the rest of them must be chosen even smaller to assure the existence of the FMNE. This observation will help us in obtaining the asymptotic distribution of S_n in the heterogeneous case.

For the rest of the paper, we focus on the asymptotic distribution of S_n at the FMNE of $G(n, \mathbf{c}_n)$. Slightly more general results for any sequence of Nash equilibria at which the number of selfish transmitters contending for channel access goes to infinity can be found in [15].

B. Limiting Behavior of the Costs of Transmitters at the FMNE

Our next result will show that there can be only one accumulation point $c \in (0, \infty)$ of the costs of the transmitters contending for the access of the common wireless channel when the number of selfish transmitters goes to infinity. Set $a_i = \frac{c_i}{1+c_i}$ and

$$a_{\min}(n) = \min_{1 \leq i \leq n} a_i.$$

We will assume that the costs of unsuccessful transmission of the nodes depend only on their internal parameters such as remaining battery lifetime or energy spent per transmission. Therefore, adding new transmitters to the game does not change the costs of the transmitters already playing the game. Thus, the limit

$$\alpha = \inf_{i \geq 1} a_i = \lim_{n \to \infty} a_{\min}(n)$$

is well-defined.

The following two results will help in proving the main theorem, Theorem 4, of the paper. The first one states the convergence of the geometric mean of the numbers $a_1, a_2, \ldots, a_n$ to a constant $\alpha > 0$ as $n \to \infty$ if the FMNE exists for all $n \geq 2$. The second one states the convergence of the a_i's to the same constant α if the FMNE exists for all $n \geq 2$.

Lemma 1: Let $Geo(a_1, a_2, \ldots, a_n)$ denote the geometric mean of $a_1, a_2, \ldots, a_n$. If the FMNE exists for all $n \geq 2$, then

$$\lim_{n \to \infty} Geo(a_1, a_2, \ldots, a_n) = \alpha > 0.$$

Proof: Assume that the FMNE exists for all values of n greater than or equal to 2. Observe the fact that the geometric mean of a set of numbers is always greater than the minimum of this set. Using this fact and the necessary and sufficient condition for the existence of fully-mixed Nash equilibrium in Theorem 1, we have

$$Geo(a_1, a_2, \ldots, a_n) \geq a_{\min}(n) > \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n-1}}. \tag{9}$$

Let $b_n = \frac{1}{n}\sum_{i=1}^{n}\log(a_i)$, and A be a limit point of $\{b_n\}_{n\geq 2}$. Then, there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} b_{n_k} = A.$$

By (9), $A \geq \log(\alpha)$. For the same subsequence, observe that

$$\lim_{k\to\infty}\frac{1}{n_k - 1}\sum_{i=1}^{n_k}\log(a_i) = \lim_{k\to\infty}\frac{n_k}{n_k - 1} b_{n_k} = A \leq \log(\alpha).$$

Thus, all of the limit points of $\{Geo(a_1, a_2, \ldots, a_n)\}_{n\geq 2}$ are equal to α. As a result,

$$\lim_{n\to\infty} Geo(a_1, a_2, \ldots, a_n) = \alpha.$$

It is easy to see that α belongs to $[0, 1)$. Therefore, we need to show that α is strictly greater than 0 to complete the proof. Suppose now that $\alpha = 0$. Then, when n nodes play the game, the transmission probabilities of nodes 1 and 2 are given as

$$p_1(n) = 1 - a_1^{-1}\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n-1}},$$

and

$$p_2(n) = 1 - a_2^{-1}\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n-1}}.$$

The expected utility of node 1 given the event $\{X_1^{(n)} = 1\}$ can be bounded as

$$\mathbb{E}\left[u_1(\mathbf{X}^{(n)})|X_1^{(n)} = 1\right] \leq (1 - p_2(n)) - c_1 p_2(n).$$

Note that $p_2(n)$ approaches 1 as $n \to \infty$ by the supposition that $\alpha = 0$. Thus, $\mathbb{E}\left[u_1(\mathbf{X}^{(n)})|X_1^{(n)} = 1\right]$ eventually becomes negative, which contradicts the existence of the FMNE for all $n \geq 2$. ■

Theorem 2: If the FMNE exists for all $n \geq 2$, then

$$\lim_{i\to\infty} a_i = \alpha > 0. \tag{10}$$

Proof: We first show that α is a limit point of $\{a_i\}_{i=1}^{\infty}$. Suppose not. Then, there exist $N \in \mathbb{N}$ and $\epsilon > 0$ such that $a_i \geq \alpha + \epsilon$ for all $i \geq N$. For $n \geq N + 1$,

$$\begin{aligned}\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n-1}} &= \left(\prod_{i=1}^{N} a_i\right)^{\frac{1}{n-1}} \cdot \left(\prod_{i=N+1}^{n} a_i\right)^{\frac{1}{n-1}} \\ &\geq \left(\prod_{i=1}^{N} a_i\right)^{\frac{1}{n-1}} \cdot (\alpha+\epsilon)^{\frac{n-N}{n-1}}.\end{aligned}$$

Therefore, we can find $M \in \mathbb{N}$ such that $\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n-1}} \geq \alpha + \epsilon/2$ and $a_{\min}(n) < \alpha + \epsilon/2$ for all $n \geq M$. The necessary and sufficient condition for the existence of the FMNE of $G(n, \mathbf{c}_n)$ in Theorem 1 implies that the FMNE does not exist for $n \geq M$, which is a contradiction. Thus, α is a limit point of $\{a_i\}_{i=1}^{\infty}$, and in fact

$$\alpha = \liminf_{i\to\infty} a_i$$

since $\alpha = \inf_{i\geq 1} a_i$.

Now, we show that α is the only limit point of $\{a_i\}_{i=1}^{\infty}$. Suppose not, and let b be another limit point of $\{a_i\}_{i=1}^{\infty}$. Then, there exists an $\epsilon > 0$ such that $b - 2\epsilon > \alpha + \epsilon$. Let

$$g_\epsilon(n) : \mathbb{N} \mapsto \mathbb{N}$$

be the function showing the number of a_i's belonging to $(b - \epsilon, b + \epsilon)$ among the first n a_i's. Since b is a limit point of $\{a_i\}_{i=1}^{\infty}$, we have

$$\lim_{n\to\infty} g_\epsilon(n) = \infty.$$

For all n large enough, we have $a_{\min}(n) \in [\alpha, \alpha + \epsilon)$. Then,

$$\begin{aligned}\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n-1}} &\geq (a_{\min}(n))^{\frac{n-g_\epsilon(n)}{n-1}} \cdot (a_{\min}(n) + \epsilon)^{\frac{g_\epsilon(n)}{n-1}} \\ &= a_{\min}(n)\,(a_{\min}(n))^{\frac{1}{n-1}} \cdot \left(1 + \frac{\epsilon}{a_{\min}(n)}\right)^{\frac{g_\epsilon(n)}{n-1}}.\end{aligned}$$

We now show that

$$(a_{\min}(n))^{\frac{1}{n-1}} = 1 - O\left(\frac{1}{n}\right)$$

and

$$\left(1 + \frac{\epsilon}{a_{\min}(n)}\right)^{\frac{g_\epsilon(n)}{n-1}} = 1 + O\left(\frac{g_\epsilon(n)}{n}\right).$$

We start with $\left(a_{\min}(n)\right)^{\frac{1}{n-1}}$.

$$\begin{aligned}\left|1 - a_{\min}(n)^{\frac{1}{n-1}}\right| &= \left|1 - \exp\left(\frac{\log(a_{\min}(n))}{n-1}\right)\right| \\ &= \left|-\left(\frac{\log(a_{\min}(n))}{n-1}\right) - \frac{1}{2!}\left(\frac{\log(a_{\min}(n))}{(n-1)}\right)^2 - \cdots\right| \\ &\leq \left|\frac{\log(a_{\min}(n))}{n-1}\right| \\ &\quad \cdot \left[1 + \frac{1}{2!}\left|\frac{\log(a_{\min}(n))}{(n-1)}\right| + \frac{1}{3!}\left|\frac{\log(a_{\min}(n))}{(n-1)}\right|^2 + \cdots\right] \\ &\leq \left|\frac{\log(a_{\min}(n))}{n-1}\right| \\ &\quad \cdot \left[1 + |\log(a_{\min}(n))| + \frac{1}{2!}|\log(a_{\min}(n))|^2 + \cdots\right] \\ &= \frac{1}{n-1}|\log(a_{\min}(n))| \exp\left(|\log(a_{\min}(n))|\right) \\ &\leq \frac{1}{n-1}\log\left(\frac{1}{\alpha}\right)\frac{1}{\alpha}.\end{aligned}$$

In a similar way, we prove that

$$\left(1 + \frac{\epsilon}{a_{\min}(n)}\right)^{\frac{g_\epsilon(n)}{n-1}} = 1 + O\left(\frac{g_\epsilon(n)}{n}\right).$$

$$
\begin{aligned}
&\left|1-\left(1+\frac{\epsilon}{a_{\min}(n)}\right)^{\frac{g_\epsilon(n)}{n-1}}\right| \\
&\quad=\left|1-\exp\left(\frac{g_\epsilon(n)}{n-1}\log\left(1+\frac{\epsilon}{a_{\min}(n)}\right)\right)\right| \\
&\quad=\frac{g_\epsilon(n)}{n-1}\log\left(1+\frac{\epsilon}{a_{\min}(n)}\right) \\
&\qquad\cdot\left|1+\frac{1}{2!}\left(\frac{g_\epsilon(n)}{n-1}\right)\log\left(1+\frac{\epsilon}{a_{\min}(n)}\right)\right. \\
&\qquad\left.+\frac{1}{3!}\left(\frac{g_\epsilon(n)}{n-1}\right)^2\log\left(1+\frac{\epsilon}{a_{\min}(n)}\right)^2+\cdots\right| \\
&\leq\frac{g_\epsilon(n)}{n-1}\log\left(1+\frac{\epsilon}{a_{\min}(n)}\right) \\
&\qquad\cdot\left|1+\left(\frac{g_\epsilon(n)}{n-1}\right)\log\left(1+\frac{\epsilon}{a_{\min}(n)}\right)\right. \\
&\qquad\left.+\frac{1}{2!}\left(\frac{g_\epsilon(n)}{n-1}\right)^2\log\left(1+\frac{\epsilon}{a_{\min}(n)}\right)^2+\cdots\right| \\
&\overset{(a)}{\leq}\frac{g_\epsilon(n)}{n-1}\log\left(1+\frac{\epsilon}{a_{\min}(n)}\right) \\
&\qquad\qquad\cdot\exp\left(\log\left(1+\frac{\epsilon}{a_{\min}(n)}\right)\right) \\
&\leq\frac{g_\epsilon(n)}{n-1}\left(1+\frac{\epsilon}{\alpha}\right)\log\left(1+\frac{\epsilon}{\alpha}\right),
\end{aligned}
$$

where (a) follows from observing that $g_\epsilon(n)\leq n-1$ for all n large enough. As a result,

$$
\begin{aligned}
\left(\prod_{i=1}^n a_i\right)^{\frac{1}{n-1}} &\geq a_{\min}(n)\left(1-O(\tfrac{1}{n})\right)\cdot\left(1+O\left(\tfrac{g_\epsilon(n)}{n}\right)\right) \\
&= a_{\min}(n)\left(1+O\left(\tfrac{g_\epsilon(n)}{n}\right)-O\left(\tfrac{1}{n}\right)-O\left(\tfrac{g_\epsilon(n)}{n^2}\right)\right).
\end{aligned}
$$

Eventually, $O\left(\frac{g_\epsilon(n)}{n}\right)-O\left(\frac{1}{n}\right)-O\left(\frac{g_\epsilon(n)}{n^2}\right)$ becomes positive after some $N\in\mathbb{N}$. Thus,

$$\left(\prod_{i=1}^n a_i\right)^{\frac{1}{n-1}}>a_{\min}(n)$$

for all $n\geq N$, which contradicts the existence of the FMNE for all $n\geq 2$. Therefore, α is the only limit point of $\{a_i\}_{i=1}^\infty$. ■

Having proved Theorem 2, it is not hard to prove similar convergence result about the costs of the transmitters. The following corollary to Theorem 2 establishes the desired convergence property of the costs of transmitters at the FMNE of $G(n,\mathbf{c})$.

Corollary 1: If the FMNE exists for all $n\geq 2$, then

$$\lim_{i\to\infty} c_i=c\in(0,\infty).$$

Proof: The proof follows from observing that $c_i=\frac{a_i}{1-a_i}$ and using the limiting behavior of the sequence $\{a_i\}_{i=1}^\infty$ established in Theorem 2. Note that $c=\frac{\alpha}{1-\alpha}\in(0,\infty)$ since $\alpha\in(0,1)$. ■

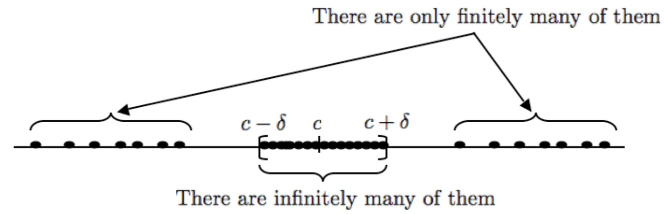

Fig. 2. An illustration of Theorem 2 and its Corollary 1. For any given arbitrarily small $\delta>0$, the costs of all the transmitters, except for at most finitely many of them, lie in the δ-neighborhood of c.

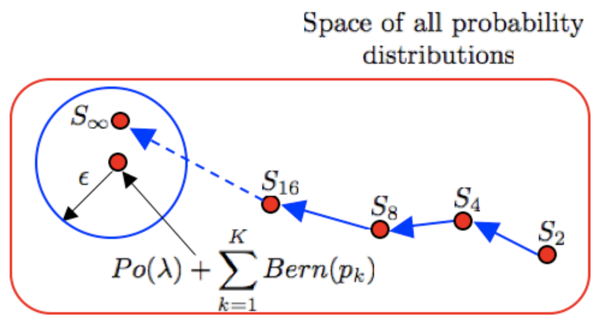

Fig. 3. A pictorial explanation of Theorem 4 . The limiting distribution of S_n lies in a small ball around the distribution of the random variable $Po(\lambda)+\sum_{k=1}^K Bern(p_k)$.

In words, Theorem 2 and its Corollary 1 say that if the FMNE exists for all $n\geq 2$, then we can find a $c>0$ such that for any given $\delta>0$, the costs of all but finitely many transmitters incurred as a result of unsuccessful transmissions are concentrated in $(c-\delta,c+\delta)$. Intuitively, we anticipate that the selfish nodes whose costs lie in $(c-\delta,c+\delta)$ to behave as in the homogeneous case. Thus, the total number of packet arrivals from these nodes can be approximated by a Poisson random variable up to an arbitrarily small error term $\epsilon(\delta)$ depending on δ. The arrivals from the other finitely many nodes whose costs lie outside of $(c-\delta,c+\delta)$ can be given by a summation of finitely many Bernoulli random variables. Therefore, we expect that once S_n converges in distribution, for any given $\epsilon>0$, we should be able to find a Poisson random variable $Po(\lambda)$ and finitely many Bernoulli random variables $\{Bern(p_j)\}_{j=1}^K$ such that S_n can be approximated, in variational distance, by the sum of $Po(\lambda)$ and $\{Bern(p_j)\}_{j=1}^K$ up to an error term less than ϵ. This observation is formally proved in Theorem 4. A pictorial representation of this fact is given in Fig. 3.

C. *Asymptotic Distribution of the Packet Arrivals at the FMNE*

We now focus on the asymptotic distribution of the packet arrivals at the FMNE. We first obtain a necessary condition for the convergence in distribution of S_n at the FMNE by using an approach based on characteristic functions. We then extend this result by obtaining a necessary and sufficient condition for the convergence in distribution of S_n at the FMNE. The proof of this extension uses concepts such as weak convergence and tightness of the probability measures, and uniform integrability (see, e.g., [17] or [18]).

In the proofs of Theorem 3 and Theorem 4, we let

$$p_{i,\infty} = \lim_{n\to\infty} p_{i,n}$$

when the FMNE exists for all $n \geq 2$. Existence of this limit can be shown by using Lemma 1 and noting that

$$p_{i,n} = 1 - a_i^{-1} \cdot \left(\prod_{j=1}^{n} a_j\right)^{\frac{1}{n-1}}$$

at the FMNE of $G(n, \mathbf{c})$.

Theorem 3: If S_n converges in distribution to a random variable S_∞, then

$$\sum_{i=1}^{\infty} p_{i,\infty} < \infty.$$

Proof: Let $\varphi_{i,n}(t)$ be the characteristic function of $X_i^{(n)}$. Then,

$$\varphi_{i,n}(t) = \mathbb{E}\big[e^{\imath t X_i^{(n)}}\big] = 1 - p_{i,n} + p_{i,n}e^{\imath t},$$

where $\imath^2 = -1$. Let $\varphi_n(t)$ and $\varphi_\infty(t)$ be characteristic functions of S_n and S_∞, respectively. Then, by the continuity theorem (see [18]), $\varphi_n(t) \to \varphi_\infty(t)$ as $n \to \infty$ for all $t \in (-\infty, \infty)$.

Now, we will show that there exist $t_0 > 0$ and $\epsilon > 0$ such that $|\varphi_\infty(t_0)| > 0$ and $|\varphi_{i,n}(t_0)|^2 \leq 1 - \epsilon p_{i,n}(1-p_{i,n})$. Since $\varphi_\infty(t)$ is a continuous function and $|\varphi_\infty(0)| = 1$, there exists a $\delta > 0$ such that $|\varphi_\infty(t)| > 0$ for all $t \in (0, \delta)$. Set $f(t) = |\varphi_{i,n}(t)|^2$. Then, it can be shown that

$$f^{(2m)}(t) = (-1)^m 2(1-p_{i,n})p_{i,n}\cos(t),$$

and

$$f^{(2m+1)}(t) = (-1)^{m+1} 2(1-p_{i,n})p_{i,n}\sin(t),$$

where $f^{(k)}(t)$ represents the kth derivative of $f(t)$ with respect to t.

By Taylor's theorem, we have

$$\begin{aligned} f(t) &= \sum_{m=0}^{\infty} f^{(m)}(0)\frac{t^m}{m!} \\ &= 1 + \sum_{m=1}^{\infty} (-1)^m 2(1-p_{i,n})p_{i,n}\frac{t^{2m}}{(2m)!} \\ &= 1 - (1-p_{i,n})p_{i,n}\left(t^2 + O(t^4)\right) \\ &\leq 1 - \frac{t^2}{2}(1-p_{i,n})p_{i,n} \text{ for sufficiently small } t. \end{aligned}$$

We choose a $t_0 \in (0, \delta)$ small enough that

$$f(t_0) \leq 1 - \frac{t_0^2}{2}(1-p_{i,n})p_{i,n}.$$

Put $\epsilon = \frac{t_0^2}{2}$. Therefore, we obtain $f(t_0) \leq 1 - \epsilon(1-p_{i,n})p_{i,n}$, and ϵ does not depend on $p_{i,n}$. For these ϵ and t_0, we have

$$\begin{aligned} -\infty &< \log\left(|\varphi_\infty(t_0)|^2\right) \\ &= \lim_{n\to\infty} \log\left(|\varphi_n(t_0)|^2\right) \\ &\leq \liminf_{n\to\infty} \sum_{i=1}^{n} \log\left(1 - \epsilon p_{i,n}(1-p_{i,n})\right). \end{aligned} \tag{11}$$

Let $s_n = -\sum_{i=1}^{n} \log\left(1 - \epsilon p_{i,n}(1-p_{i,n})\right)$. Then, all of the limit points of $\{s_n\}_{n=1}^{\infty}$ are smaller than ∞ by (11). By using Fatou's lemma, we obtain

$$\begin{aligned} \infty &> \liminf_{n\to\infty} \sum_{i=1}^{\infty} -\mathbb{1}_{\{i\leq n\}} \log\left(1 - \epsilon p_{i,n}(1-p_{i,n})\right) \\ &\geq \sum_{i=1}^{\infty} \liminf_{n\to\infty} -\mathbb{1}_{\{i\leq n\}} \log\left(1 - \epsilon p_{i,n}(1-p_{i,n})\right) \\ &= \sum_{i=1}^{\infty} -\log\left(1 - \epsilon p_{i,\infty}(1-p_{i,\infty})\right). \end{aligned}$$

Therefore, the sum

$$\sum_{i=1}^{\infty} -\log\left(1 - \epsilon p_{i,\infty}(1-p_{i,\infty})\right)$$

must be finite whenever S_n converges in distribution to another random variable S_∞. Observing $p_{i,\infty} \to 0$ as $i \to \infty$ by means of Lemma 1 and Theorem 2, and using L'Hopital's rule, one can show

$$\lim_{i\to\infty} \frac{-\log\left((1 - \epsilon p_{i,\infty}(1-p_{i,\infty}))\right)}{p_{i,\infty}} = \epsilon \in (0, \infty).$$

As a result, we also conclude that $\sum_{i=1}^{\infty} p_{i,\infty} < \infty$. ■

We now extend Theorem 3 by proving a necessary and sufficient condition for the convergence in distribution of S_n in distribution at the FMNE as the number of selfish transmitters contending for the channel access goes to infinity. We also obtain the asymptotic form of the packet arrival distribution in Theorem 4, which formally confirms the intuitive explanation regarding the structure of the asymptotic packet arrival distribution given in the previous section. Before going into the details of the proof of this result, we first would like to discuss one technical difficulty which makes the proof complicated.

Recall that the action, $X_i^{(n)}$, chosen by the ith transmitter at the FMNE is distributed according to $Bern(p_{i,n})$ when there are n transmitters contending for the channel access. Then, the total number packet arrivals is equal to $S_n = \sum_{i=1}^{n} X_i^{(n)}$, and its expectation is equal to $\mathbb{E}[S_n] = \sum_{i=1}^{n} p_{i,n}$. In general, one cannot claim that

$$\lim_{n\to\infty} \sum_{i=1}^{n} p_{i,n} = \sum_{i=1}^{\infty} p_{i,\infty}$$

because the dominated convergence theorem cannot be justified. For example, consider the homogenous case in which all transmitters have identical utility functions. In this case, we have

$$p_{i,n} = 1 - \left(\frac{c}{1+c}\right)^{\frac{1}{n-1}}$$

for all transmitters $i \in \mathcal{N}$. Then,

$$p_{i,\infty} = \lim_{n\to\infty} 1 - \left(\frac{c}{1+c}\right)^{\frac{1}{n-1}} = 0,$$

and therefore, $\sum_{i=1}^{\infty} p_{i,\infty} = 0$. However,

$$\lim_{n\to\infty} \sum_{i=1}^{n} p_{i,n} = -\log\left(\frac{c}{1+c}\right). \tag{12}$$

Thus, one needs to be careful in the proof of Theorem 4 while interchanging the order of limits and sums.

The following lemma will be helpful during the proof of Theorem 4. We also provide its proof for the sake of completeness.

Lemma 2: Let $\{\mu_i\}_{i=1}^n$ and $\{\nu_i\}_{i=1}^n$ be two collections of probability distributions on $\mathbb{Z}$ and $n \geq 2$. Let $\mu = \mu_1 * \mu_2 * \cdots * \mu_n$ and $\nu = \nu_1 * \nu_2 * \cdots * \nu_n$ denote convolutions of these probability distributions. Then,

$$d_V(\mu, \nu) \leq \sum_{i=1}^{n} d_V(\mu_i, \nu_i). \tag{13}$$

Proof: It is enough to prove this lemma for $n = 2$. The general case for n greater than 2 follows from the repeated applications of this result by considering μ_1 (ν_1) and $\mu_2 * \mu_3 * \cdots * \mu_n$ ($\nu_2 * \nu_3 * \cdots * \nu_n$) separately. By recalling the commutativity property of convolution of probability distributions, we have

$$\begin{aligned} d_V(\mu_1 * \mu_2, \nu_1 * \nu_2) &\leq d_V(\mu_1 * \mu_2, \nu_1 * \mu_2) \\ &\quad + d_V(\mu_2 * \nu_1, \nu_2 * \nu_1) \\ &= \sum_{z \in \mathbb{Z}} |\mu_1 * \mu_2(z) - \nu_1 * \mu_2(z)| \\ &\quad + \sum_{z \in \mathbb{Z}} |\mu_2 * \nu_1(z) - \nu_2 * \nu_1(z)|. \end{aligned}$$

Consider the first summation.

$$\begin{aligned} &\sum_{z \in \mathbb{Z}} |\mu_1 * \mu_2(z) - \nu_1 * \mu_2(z)| \\ &= \sum_{z \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} \mu_1(k)\mu_2(z-k) - \nu_1(k)\mu_2(z-k) \right| \\ &\leq \sum_{z \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \mu_2(z-k) \cdot |\mu_1(k) - \nu_1(k)| \\ &= \sum_{k \in \mathbb{Z}} |\mu_1(k) - \nu_1(k)| \cdot \sum_{z \in \mathbb{Z}} \mu_2(z-k) \\ &= d_V(\mu_1, \nu_1). \end{aligned}$$

Similarly,

$$\sum_{z \in \mathbb{Z}} |\mu_2 * \nu_1(z) - \nu_2 * \nu_1(z)| \leq d_V(\mu_2, \nu_2).$$

As a result,

$$d_V(\mu_1 * \mu_2, \nu_1 * \nu_2) \leq d_V(\mu_1, \nu_1) + d_V(\mu_2, \nu_2). \tag{14}$$

■

Theorem 4: Assume the FMNE exists for all $n \geq 2$. Then, S_n converges in distribution if and only if

$$\lim_{n \to \infty} \sum_{i=1}^{n} p_{i,n} = m \in (0, \infty).$$

Moreover, whenever S_n converges in distribution, for any $\epsilon > 0$, there exists a Poisson random variable $Po(\lambda)$ and a collection of finitely many Bernoulli random variables $\{Bern(p_k)\}_{k=1}^K$ such that

$$\limsup_{n \to \infty} d_V\left(S_n, Po(\lambda) + \sum_{k=1}^{K} Bern(p_k)\right) \leq \epsilon. \tag{15}$$

Proof: $\Longleftarrow$: We first show the *if* direction. Suppose

$$\lim_{n \to \infty} \sum_{i=1}^{n} p_{i,n} = m \in (0, \infty)$$

exists. Let $m_n = \sum_{i=1}^n p_{i,n}$ and S_n be distributed according to μ_n. We will first show that $\{\mu_n\}_{n=1}^\infty$ is a tight sequence of distributions. To this end, we show that for each $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that $\mathbb{P}\{S_n \in [0, M]\} \geq 1 - \epsilon$. Choose a $\delta > 0$ and choose $N \in \mathbb{N}$ large enough that $m_n \in [m - \delta, m + \delta]$ for all $n \geq N$. Then, by the Markov inequality,

$$\mathbb{P}\{S_n > M\} \leq \frac{\mathbb{E}\left[(S_n - m_n)^2\right]}{(M - m_n)^2}.$$

We bound $\mathbb{E}\left[(S_n - m_n)^2\right]$ as follows:

$$\begin{aligned} \mathbb{E}\left[(S_n - m_n)^2\right] &= \sum_{i=1}^{n} Var\left(X_i^{(n)}\right) \\ &\leq m_n \leq m + \delta. \end{aligned}$$

In addition, $(M - m_n)^2 \geq (M - m - \delta)^2$. Thus,

$$\mathbb{P}\{S_n > M\} \leq \frac{m + \delta}{(M - m - \delta)^2}.$$

If M is large enough, then we have $\mathbb{P}\{S_n > M\} \leq \epsilon$ for all $n \geq N$. By making M larger, if necessary, we have $\mathbb{P}\{S_n > M\} = 0$ for all $n < N$. As a result,

$$\mathbb{P}\{S_n \in [0, M]\} > 1 - \epsilon \text{ for all } n.$$

Thus, $\{\mu_n\}_{n=1}^\infty$ is a tight sequence of distributions.

Now, we will show that μ_n converges, in variational distance, to a distribution μ. This fact, combined with the tightness of $\{\mu_n\}_{n=1}^\infty$, will imply that μ is in fact a probability distribution and $\mu_n \Rightarrow \mu$. By using Lemma 1 and Theorem 2, it is easy to see that

$$\lim_{i \to \infty} p_{i,\infty} = 0.$$

Thus, for any given $\epsilon > 0$, we can choose K large enough that

$$\max_{i \geq K} p_{i,\infty} \leq \frac{\epsilon}{8m}.$$

Let $\lambda_n = \sum_{i=K}^n p_{i,n}$ and $\lambda = \lim_{n \to \infty} \lambda_n$. Then, by using Lemma 2, $d_V\left(S_n, Po(\lambda) + \sum_{i=1}^{K-1} Bern(p_{i,\infty})\right)$ can be bounded above as

$$\begin{aligned} &d_V\left(S_n, Po(\lambda) + \sum_{i=1}^{K-1} Bern(p_{i,\infty})\right) \\ &\leq d_V\left(\sum_{i=1}^{K-1} X_i^{(n)}, \sum_{i=1}^{K-1} Bern(p_{i,\infty})\right) \\ &\quad + d_V\left(\sum_{i=K}^{n} X_i^{(n)}, Po(\lambda)\right). \end{aligned}$$

Let us now focus on $d_V\left(\sum_{i=K}^n X_i^{(n)}, Po(\lambda)\right)$. Since $d_V(\cdot, \cdot)$ forms a metric on the space of all discrete probability distributions on $\mathbb{Z}$, we can write

$$d_V\left(\sum_{i=K}^{n} X_i^{(n)}, Po(\lambda)\right) \le d_V\left(\sum_{i=K}^{n} X_i^{(n)}, Po(\lambda_n)\right) + d_V\left(Po(\lambda_n), Po(\lambda)\right).$$

By using Le Cam's inequality,

$$d_V\left(\sum_{i=K}^{n} X_i^{(n)}, Po(\lambda_n)\right) \le 2\sum_{i=K}^{n} p_{i,n}^2. \tag{16}$$

Thus,

$$\begin{aligned} d_V&\left(S_n, Po(\lambda) + \sum_{i=1}^{K-1} Bern(p_{i,\infty})\right) \\ &\le d_V\left(\sum_{i=1}^{K-1} X_i^{(n)}, \sum_{i=1}^{K-1} Bern(p_{i,\infty})\right) \\ &\quad + d_V\left(Po(\lambda_n), Po(\lambda)\right) + 2\sum_{i=K}^{n} p_{i,n}^2 \\ &\le d_V\left(\sum_{i=1}^{K-1} X_i^{(n)}, \sum_{i=1}^{K-1} Bern(p_{i,\infty})\right) \\ &\quad + d_V\left(Po(\lambda_n), Po(\lambda)\right) + 2\max_{K\le i\le n} p_{i,n} \sum_{i=K}^{n} p_{i,n}. \end{aligned}$$

By noting that $\sum_{i=1}^{K-1} X_i^{(n)} \Rightarrow \sum_{i=1}^{K-1} Bern(p_{i,\infty})$ and $Po(\lambda_n) \Rightarrow Po(\lambda)$ as n goes to infinity, we obtain

$$\begin{aligned} \limsup_{n\to\infty} & d_V\left(S_n, \sum_{i=1}^{K-1} Bern(p_{i,\infty}) + Po(\lambda)\right) \\ &\le 2\limsup_{n\to\infty}\left(\lambda_n \cdot \max_{K\le i\le n} p_{i,n}\right) \\ &= 2\lambda \limsup_{n\to\infty} \max_{K\le i\le n} p_{i,n} \quad (\text{since } \lambda_n \to \lambda). \end{aligned}$$

Let $i(n)$ be such that $p_{i(n),n} = \max_{K\le i\le n} p_{i,n}$. Then, there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} p_{i(n_k),n_k} = \limsup_{n\to\infty} \max_{K\le i\le n} p_{i,n}.$$

If $\{i(n_k)\}_{k=1}^{\infty}$ is a bounded sequence, there exists a further subsequence $\{i(n_{k_j}\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} i(n_{k_j}) = i^{**}$. Since we are considering a sequence of integers converging to another integer, there exists $N \in \mathbb{N}$ such that we have $i(n_{k_j}) = i^{**}$ for all $j \ge N$. Thus,

$$\limsup_{n\to\infty} \max_{K\le i\le n} p_{i,n} = p_{i^{**},\infty} \le \max_{i\ge K} p_{i,\infty}.$$

If $\{i(n_k)\}_{k=1}^{\infty}$ is not a bounded sequence, then there exists a further subsequence $\{i(n_{k_j})\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} i(n_{k_j}) = \infty$. Let $\gamma_n = \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n-1}}$. Recall that transmission probabilities at the FMNE are given as $p_{i,n} = 1 - a_i^{-1}\gamma_n$. So,

$$\begin{aligned} \limsup_{n\to\infty} \max_{K\le i\le n} p_{i,n} &= \lim_{j\to\infty} p_{i(n_{k_j}),n_{k_j}} \\ &= 1 - \lim_{j\to\infty} a_{i(n_{k_j})}^{-1} \lim_{j\to\infty} \gamma_{n_{k_j}} \\ &= 0 \le \max_{i\ge K} p_{i,\infty}. \end{aligned}$$

Therefore,

$$\begin{aligned} \limsup_{n\to\infty} & d_V\left(S_n, Po(\lambda) + \sum_{i=1}^{K-1} Bern(p_{i,\infty})\right) \\ &\le 2\lambda \max_{i\ge K} p_{i,\infty} \le \frac{\epsilon}{4}. \end{aligned}$$

Thus, there exists $N \in \mathbb{N}$ large enough so that

$$d_V\left(S_n, Po(\lambda) + \sum_{i=1}^{K-1} Bern(p_{i,\infty})\right) \le \frac{\epsilon}{2}$$

for all $n \ge N$. As a result, $\{\mu_n\}_{n=1}^{\infty}$ is a Cauchy sequence with respect to the metric d_V on the set of all probability measures $\mathcal{Z}$ on $\mathbb{Z}$. That is, $d_V(\mu_n, \mu_m) \le \epsilon$ for all $n, m \ge N$. This also implies that $\{\mu_n(z)\}_{n=1}^{\infty}$ is a Cauchy sequence for all $z \in \mathbb{Z}$, and therefore, converges for any $z \in \mathbb{Z}$. Let $\mu(z) = \lim_{n\to\infty} \mu_n(z)$ for all $z \in \mathbb{Z}$. This, combined with the tightness of $\{\mu_n\}_{n=1}^{\infty}$, implies that μ is a probability measure and $\mu_n \Rightarrow \mu$.

$\Longrightarrow$: Now, we prove the *only if* part. In fact, this will be a general result for any sequence of triangular arrays of Bernoulli random variables. Suppose now that there exists an $\mathbb{R}$ valued random variable S_∞ such that S_n converges in distribution to S_∞. First, assume

$$\limsup_{n\to\infty} m_n = \infty,$$

and let $Y_i^{(n)} = X_i^{(n)} - p_{i,n}$. Set $R_n = \sum_{i=1}^{n} Y_i^{(n)}$. Consider $\mathbb{E}\left[e^{-tY_i^{(n)}}\right]$ for $t > 0$. We have

$$\begin{aligned} \mathbb{E}&\left[e^{-tY_i^{(n)}}\right] \\ &= 1 + \frac{1}{2!}t^2\mathbb{E}\left[\left(Y_i^{(n)}\right)^2\right] + \frac{1}{3!}(-t)^3\mathbb{E}\left[\left(Y_i^{(n)}\right)^3\right] + \cdots \\ &\le 1 + \frac{1}{2!}t^2\left|\mathbb{E}\left[\left(Y_i^{(n)}\right)^2\right]\right| + \frac{1}{3!}t^3\left|\mathbb{E}\left[\left(Y_i^{(n)}\right)^3\right]\right| + \cdots. \end{aligned}$$

We will show $\left|\mathbb{E}\left[\left(Y_i^{(n)}\right)^k\right]\right| \le p_{i,n}$ for all k. For $k = 2$,

$$\begin{aligned} \mathbb{E}\left[\left(Y_i^{(n)}\right)^2\right] &= Var\left(X_i^{(n)}\right) \\ &\le \mathbb{E}\left[\left(X_i^{(n)}\right)^2\right] = p_{i,n}. \end{aligned}$$

For any $k \ge 3$, we have

$$\begin{aligned} &\left|\mathbb{E}\left[\left(Y_i^{(n)}\right)^k\right]\right| \\ &\le \mathbb{E}\left[\left|Y_i^{(n)}\right|^{k-1} \cdot \left|Y_i^{(n)}\right|\right] \\ &\le \left(\mathbb{E}\left[\left|Y_i^{(n)}\right|^{2k-2}\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[\left|Y_i^{(n)}\right|^2\right]\right)^{\frac{1}{2}} \quad \text{(Hölder's Ineq.)} \\ &\le \left(\mathbb{E}\left[\left|Y_i^{(n)}\right|^2\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[\left|Y_i^{(n)}\right|^2\right]\right)^{\frac{1}{2}} \le p_{i,n}. \end{aligned}$$

Thus,

$$\mathbb{E}\left[e^{-tY_i^{(n)}}\right] \le 1 + \frac{1}{2!}t^2 p_{i,n} + p_{i,n}\left(\frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right). \tag{17}$$

Then, there is a $\delta_1 > 0$ such that, uniformly over all $p_{i,n}$, we have

$$\mathbb{E}\left[e^{-tY_i^{(n)}}\right] \quad \leq \quad 1 + t^2 p_{i,n} \quad \text{for all } t \in (0, \delta_1).$$

Now, make δ_1 smaller (if necessary) so that $t^2 \leq \frac{t}{4}$. Then, for $t \in (0, \delta_1)$, we have

$$\begin{aligned}
&\mathbb{P}\left\{S_n \leq \frac{m_n}{2}\right\} \\
&\quad = \mathbb{P}\left\{e^{\frac{-t}{2}R_n} \geq e^{\frac{tm_n}{4}}\right\} \\
&\quad \leq \frac{\mathbb{E}\left[e^{-tR_n}\right]}{e^{\frac{tm_n}{2}}} \quad \text{(Markov Inequality)} \\
&\quad = e^{\frac{-tm_n}{2}} \prod_{i=1}^{n} \mathbb{E}\left[e^{-tY_i^{(n)}}\right] \\
&\quad \leq e^{\frac{-tm_n}{2}} \prod_{i=1}^{n} (1 + t^2 p_{i,n}) \\
&\quad = \exp\left(\sum_{i=1}^{n}\left(\frac{-tp_{i,n}}{2} + \log\left(1 + t^2 p_{i,n}\right)\right)\right) \\
&\quad \leq \exp\left(\sum_{i=1}^{n}\left(\frac{-tp_{i,n}}{2} + t^2 p_{i,n}\right)\right) \quad \text{(since } \log(x) \leq x - 1\text{)} \\
&\quad \leq \exp\left(\frac{-t}{4}\sum_{i=1}^{n} p_{i,n}\right).
\end{aligned}$$

Since $\limsup_{n\to\infty} m_n = \infty$, we can find a subsequence of $\{m_n\}_{n=1}^{\infty}$, which we call $\{m_{n(k)}\}_{k=1}^{\infty}$, such that $m_{n(k)} \geq k$, $\forall k$. Then, $\mathbb{P}\left\{S_{n(k)} \leq \frac{m_{n(k)}}{2}\right\} \leq \exp\left(\frac{-tk}{4}\right)$. On setting

$$\mathcal{A}_k = \{\omega \in \Omega : S_{n(k)}(\omega) \leq \frac{m_{n(k)}}{2}\},$$

we have

$$\sum_{k=1}^{\infty} \mathbb{P}\left(\mathcal{A}_k\right) < \infty.$$

By the Borel-Cantelli lemma, $\mathbb{P}\{\mathcal{A}_k \ i.o\} = 0$. Thus,

$$\lim_{k\to\infty} S_{n(k)} = \infty. \text{ w.p.1.}$$

Since almost sure convergence implies convergence in distribution, we have $S_\infty = \infty$ w.p.1, which is a contradiction. Thus,

$$\limsup_{n\to\infty} m_n < \infty.$$

In this case, we can find $C < \infty$ such that $m_n \leq C$ for all n. Now, we will show that $\{S_n\}_{n=1}^{\infty}$ is uniformly integrable.

$$\begin{aligned}
\mathbb{E}\left[S_n^2\right] &= \mathbb{E}\left[\sum_{i=1}^{n}\left(X_i^{(n)}\right)^2\right] + \mathbb{E}\left[\sum_{\substack{i,j=1 \\ i\neq j}}^{n} X_i^{(n)} X_j^{(n)}\right] \\
&\leq \sum_{i=1}^{n} p_{i,n} + \sum_{i,j=1}^{n} p_{i,n} p_{j,n} \\
&= m_n + m_n^2 \leq C\left(1 + C\right) < \infty.
\end{aligned}$$

Therefore, $\{S_n\}_{n=1}^{\infty}$ is uniformly integrable. By Skorohod's representation theorem (see [17]), there exists a probability space $(\Omega', \mathcal{F}', P')$ and random variables S'_n and S'_∞ having the same distributions as S_n and S_∞, respectively, such that $S'_n \to S'_\infty$ w.p.1. By uniform integrability, we also have L^1 convergence, i.e.,

$$\lim_{n\to\infty} \mathbb{E}\left[S'_n\right] = \mathbb{E}\left[S'_\infty\right]. \tag{18}$$

As a result, the limit $\lim_{n\to\infty} m_n$ exists and belongs to $(0, \infty)$. ■

IV. Conclusion

In this paper, we have analyzed the asymptotic behavior of multiple-access networks containing large numbers of selfish transmitters that share a common wireless communication channel to communicate with their intended receivers. In particular, we have focused on the asymptotic distribution of the total number of packet arrivals to the common wireless channel coming from these selfish transmitters.

When selfish transmitters are identical to one another in their utility functions, the asymptotic distribution of the total number of packet arrivals becomes a Poisson distribution. On the other hand, when selfish transmitters do not have identical utility functions, we have first obtained a necessary and sufficient condition for the total number of packet arrivals to converge in distribution. We then have shown that the asymptotic packet arrival distribution can be arbitrarily closely approximated in distribution by a summation of finitely many independent Bernoulli random variables and an independent Poisson random variable.

From the practical point of view for modeling and analyzing multiple-access communication networks, this result further implies that assuming packet arrivals to a common wireless communication channel to be governed by a pure Poisson process is not correct for multiple-access communication networks consisting of selfish transmitters with different utility functions. Rather, one should assume that the distribution of packet arrivals is a mixture of a Poisson distribution and finitely many Bernoulli distributions while modeling and analyzing such networks.

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