

POWER CONTROL BY GEOMETRIC PROGRAMMING

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Abstract. In wireless cellular or ad hoc networks that are interference-limited, a variety of power control problems can be formulated as nonlinear optimization with a system-wide objective, *e.g.*, maximizing total system throughput or achieving maxmin fairness, and many QoS constraints from individual users, *e.g.*, on data rate, delay, and outage probability. We show that in the high SIR regime, these nonlinear and apparently difficult, nonconvex optimization problems can be transformed into convex optimization problems through geometric programming, thus can be very efficiently solved for global optimality even in large networks. In the medium to low SIR regime, these constrained nonlinear optimization of power control cannot be turned into tractable convex formulations, but a heuristics can be used to compute the optimal solution by solving a series of geometric programs. These techniques for power control, together with their implications to admission control and pricing in wireless networks, are illustrated through several numerical examples.

1. Introduction. Due to the broadcast nature of radio transmission, data rates in a wireless network are affected by signal interference. This is particularly important in CDMA systems where users transmit at the same time over the same frequency bands and their spreading codes are not perfectly orthogonal. Transmit power control is often used to tackle this problem of signal interference. In this chapter, we study how to optimize over the transmit powers to create the optimal set of Signal-to-Interference Ratios (SIR) on wireless links. Optimality here may be referring to maximizing a system-wide efficiency metric (*e.g.*, the total system throughput), or maximizing a Quality of Service (QoS) metric for a user in the highest QoS class, or maximizing a QoS metric for the user with the minimum QoS metric value (*i.e.*, a maxmin optimization).

While the objective represents a system-wide goal to be optimized, individual users' QoS requirements must also be satisfied. Any power allocation must therefore be constrained by a feasible set formed by these minimum requirements from the users. Such a constrained optimization captures the tradeoff between user-centric constraints and network-centric objective. Because a higher power level from one transmitter increases the interference levels at other receivers, there may not be any feasible power allocation to satisfy the requirements from all the users. Sometimes an existing set of requirements can be satisfied, but when a new user is admitted into the system, there exists no more feasible power control solutions, or the maximized objective is reduced due to the tightening of the constraint set, leading to the need for admission control and admission pricing.

Because many QoS metrics are nonlinear functions of SIR, which is in turn a nonlinear (and neither convex nor concave) function of transmit powers, the above power control problems are difficult nonlinear optimization problems that may appear to be not efficiently solvable. This chapter shows that, when SIR is much larger than 0dB, a class of nonlinear optimization called Geometric Programming (GP) can be used to efficiently compute the globally optimal power control in many of these problems, and efficiently determine the feasibility of user requirements by returning either a feasible (and indeed optimal) set of powers or a certificate of infeasibility. This leads to an effective admission control and admission pricing method. The

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key observation is that despite the apparent nonconvexity, GP technique turns these constrained optimization of power control into nonlinear yet still convex optimization, which is intrinsically tractable despite its nonlinearity in objective and constraints. When SIR is comparable to or below 0dB, the power control problems are ‘truly’ nonconvex. In this case, we present a heuristics that often computes the globally optimal power allocation by solving a sequence of GPs.

Power control by GP is applicable to formulations in both cellular networks with single-hop transmission between mobile users and base stations and ad hoc networks with multihop transmission among the nodes, as illustrated through several numerical examples in this chapter. Traditionally, GP is solved by centralized computation through the highly efficient interior point methods. The appendix in this chapter outlines how GP can also be solved distributively with message passing.

2. Geometric Programming. GP is a class of nonlinear, nonconvex optimization with many useful theoretical and computational properties. Since a GP can be turned into a convex optimization problem ¹, a local optimum is also a global optimum, duality gap is zero under mild conditions, and a global optimum can be computed very efficiently. Numerical efficiency holds both in theory and in practice: interior point methods applied to GP have provably polynomial time complexity [18], and very fast in practice (see, *e.g.*, the algorithms and discussions on numerical efficiency in [16]) with high-quality software downloadable from the Internet (*e.g.*, the MOSEK package). Convexity and duality properties of GP are well understood, and large-scale, robust numerical solvers for GP are available. Furthermore, special structures in GP and its Lagrange dual problem lead to distributed algorithms, physical interpretations, and computational acceleration beyond the generic results for convex optimization.

GP was invented in 1960s [12] and applied to primarily mechanical and chemical engineering problems in 1960s and 1970s and then to several other science and engineering disciplines [1, 3, 4, 5]. Since mid-1990s, GP has been used to solve a variety of analysis and design problems in communication systems, including recently to wireless network power control [14, 15], resource allocation [9], joint congestion control and power control [6], queuing systems [10], information theory [8], as well as to coding and signal processing. A detailed tutorial of GP and comprehensive survey of its recent applications to communication systems can be found in [7]. This section contains a brief introduction of GP terminology for applications to be shown in the next two sections.

2.1. Basic formulations. There are two equivalent forms of GP: standard form and convex form. The first is a constrained optimization of a type of function called posynomial, and the second form is obtained from the first through logarithmic change of variable. ²

We first define a monomial as a function $f : \mathbf{R}_{++}^n \rightarrow \mathbf{R}$:

$$f(\mathbf{x}) = dx_1^{a^{(1)}} x_2^{a^{(2)}} \dots x_n^{a^{(n)}}$$

where the multiplicative constant $d \geq 0$ and the exponential constants $a^{(j)} \in \mathbf{R}, j =$

¹Minimizing a convex objective function subject to upper bound inequality constraints on convex constraint functions and linear equality constraints.

²Standard form GP is often used in network resource allocation problems, and convex form GP in problems based on stochastic models such as information theoretic problems.

$1, 2, \dots, n$. A sum of monomials, indexed by k below, is called a posynomial:

$$f(\mathbf{x}) = \sum_{k=1}^K d_k x_1^{a_k^{(1)}} x_2^{a_k^{(2)}} \dots x_n^{a_k^{(n)}}.$$

where $d_k \geq 0$, $k = 1, 2, \dots, K$, and $a_k^{(j)} \in \mathbf{R}$, $j = 1, 2, \dots, n, k = 1, 2, \dots, K$. The key features about posynomial are its positivity and convexity (in log domain).

For example, $2x_1^{-\pi}x_2^{0.5} + 3x_1x_3^{100}$ is a posynomial in \mathbf{x} , $x_1 - x_2$ is not a posynomial, and x_1/x_2 is a monomial, thus also a posynomial.

Minimizing a posynomial subject to posynomial upper bound inequality constraints and monomial equality constraints is called GP in standard form:

$$(2.1) \quad \begin{aligned} & \text{minimize} && f_0(\mathbf{x}) \\ & \text{subject to} && f_i(\mathbf{x}) \leq 1, \quad i = 1, 2, \dots, m, \\ & && h_l(\mathbf{x}) = 1, \quad l = 1, 2, \dots, M \end{aligned}$$

where f_i , $i = 0, 1, \dots, m$, are posynomials: $f_i(\mathbf{x}) = \sum_{k=1}^{K_i} d_{ik} x_1^{a_{ik}^{(1)}} x_2^{a_{ik}^{(2)}} \dots x_n^{a_{ik}^{(n)}}$, and h_l , $l = 1, 2, \dots, M$ are monomials: $h_l(\mathbf{x}) = d_l x_1^{a_l^{(1)}} x_2^{a_l^{(2)}} \dots x_n^{a_l^{(n)}}$.³

GP in standard form is not a convex optimization problem, because posynomials are not convex functions. However, with a logarithmic change of the variables and multiplicative constants: $y_i = \log x_i$, $b_{ik} = \log d_{ik}$, $b_l = \log d_l$, we can turn it into the following equivalent problem⁴ in \mathbf{y} :

$$(2.2) \quad \begin{aligned} & \text{minimize} && p_0(\mathbf{y}) = \log \sum_{k=1}^{K_0} \exp(\mathbf{a}_{0k}^T \mathbf{y} + b_{0k}) \\ & \text{subject to} && p_i(\mathbf{y}) = \log \sum_{k=1}^{K_i} \exp(\mathbf{a}_{ik}^T \mathbf{y} + b_{ik}) \leq 0, \quad i = 1, 2, \dots, m, \\ & && q_l(\mathbf{y}) = \mathbf{a}_l^T \mathbf{y} + b_l = 0, \quad l = 1, 2, \dots, M. \end{aligned}$$

This is referred to as GP in convex form, which is a convex optimization problem since it can be verified that the log-sum-exp function is convex.

In summary, GP is a nonlinear, nonconvex optimization problem that can be transformed into a nonlinear, convex problem. Therefore, a local optimum for GP is also a global optimum, and the duality gap is zero under mild technical conditions. The Lagrange dual problem of GP has interesting structures. In particular, dual GP is linearly constrained and its objective function is a generalized entropy function [12].

Following the standard procedure of deriving the Lagrange dual problem [5], it is readily verified that for the following GP over \mathbf{y} with m posynomial constraints,

$$\begin{aligned} & \text{minimize} && \log \sum_{k=1}^{K_0} \exp(\mathbf{a}_{0k}^T \mathbf{y} + b_{0k}) \\ & \text{subject to} && \log \sum_{k=1}^{K_i} \exp(\mathbf{a}_{ik}^T \mathbf{y} + b_{ik}) \leq 0, \quad i = 1, \dots, m, \end{aligned}$$

³Note that a monomial equality constraint can also be expressed as two monomial inequality constraints: $h_l(\mathbf{x}) \geq 1$ and $h_l(\mathbf{x}) \leq 1$. Thus a standard form GP can be defined as the minimization of a posynomial under upper bound inequality constraints on posynomials.

⁴Equivalence relationship between two optimization problems is used in a loose way here. If the optimized value of problem A is a simple (*e.g.*, monotonic and invertible) function of the optimized value of problem B, and an optimizer of problem B can be easily computed from an optimizer of problem A (*e.g.*, through a simple mapping), then problems A and B are said to be equivalent.

the Lagrange dual problem is

$$\begin{aligned}
(2.3) \quad & \text{maximize} \quad \mathbf{b}_0^T \boldsymbol{\nu}_0 - \sum_{j=1}^{K_0} \nu_{0j} \log \nu_{0j} + \sum_{i=1}^m \left(\mathbf{b}_i^T \boldsymbol{\nu}_i - \sum_{j=1}^{K_i} \nu_{ij} \log \frac{\nu_{ij}}{\mathbf{1}^T \boldsymbol{\nu}_i} \right) \\
& \text{subject to} \quad \boldsymbol{\nu}_i \succeq 0, \quad i = 0, \dots, m, \\
& \quad \mathbf{1}^T \boldsymbol{\nu}_0 = 1, \\
& \quad \sum_{i=0}^m \mathbf{A}_i^T \boldsymbol{\nu}_i = 0
\end{aligned}$$

where the optimization variables are $(m+1)$ vectors: $\boldsymbol{\nu}_i$, $i = 0, 1, \dots, m$. The length of $\boldsymbol{\nu}_i$ is K_i , *i.e.*, the number of monomial terms in the i th posynomial, $i = 0, 1, \dots, m$. Here, \mathbf{A}_0 is the matrix of the exponential constants in the objective function, where each row corresponds to each monomial term (*i.e.*, \mathbf{a}_{0k}^T is the k th row in matrix \mathbf{A}_0), and \mathbf{A}_i , $i = 1, 2, \dots, m$, are the matrices of the exponential constants in the constraint functions, again with each row corresponding to each monomial term. The multiplicative constants in the objective function are denoted as \mathbf{b}_0 and those in the i th constraint as \mathbf{b}_i , $i = 1, 2, \dots, m$. Linearity of dual problem constraints is utilized in some very efficient GP solvers (*e.g.*, [16]) that solve both the primal and dual problems of a GP simultaneously.

2.2. Feasibility and sensitivity analysis. Testing whether there is any variable \mathbf{x} that satisfies a set of posynomial inequality and monomial equality constraints:

$$(2.4) \quad f_i(\mathbf{x}) \leq 1, \quad i = 1, \dots, m, \quad h_l(\mathbf{x}) = 1, \quad l = 1, \dots, M$$

is called a GP feasibility problem. Solving feasibility problem is useful when we would like to determine whether the constraints are too tight to allow any feasible solution, or when it is necessary to generate a feasible solution as the initial point of an interior-point algorithm.

Feasibility of the monomial equality constraints can be verified by checking feasibility of the linear system of equations that the monomial constraints get logarithmically transformed into. Feasibility of the posynomial inequality constraints can then be verified by solving the following GP, introducing an auxiliary variable $s \in \mathbf{R}$ in addition to variables $\mathbf{x} \in \mathbf{R}^n$ [12, 4]:

$$\begin{aligned}
(2.5) \quad & \text{minimize} \quad s \\
& \text{subject to} \quad f_i(\mathbf{x}) \leq s, \quad i = 1, \dots, m \\
& \quad g_l(\mathbf{x}) = 1, \quad l = 1, \dots, M, \\
& \quad s \geq 1.
\end{aligned}$$

This GP always has a feasible solution: $s = \max\{1, \max_i \{f_i(\mathbf{x})\}\}$ for any \mathbf{x} that satisfies the monomial equality constraints. Now solve problem (2.5) and obtain the optimal (s^*, \mathbf{x}^*) . If $s^* = 1$, then the set of posynomial constraints $f_i(\mathbf{x}) \leq 1$ are feasible, and the associated \mathbf{x}^* is a feasible solution to the original feasibility problem (2.4). Otherwise, the set of posynomial constraints is infeasible.

The constant parameters in a GP may be based on inaccurate estimates or vary over time. As constant parameters change a little, we may not want to solve the slightly perturbed GP from scratch. It is useful to directly determine the impact of small perturbations of constant parameters on the optimal solution. Suppose we loosen the i th inequality constraint (with $u_i > 1$) or tighten it (with $u_i < 1$), and

shift the j th equality constraint (with $v_j \in \mathbf{R}$):

$$(2.6) \quad \begin{aligned} & \text{minimize} && f_0(\mathbf{x}) \\ & \text{subject to} && f_i(\mathbf{x}) \leq u_i, \quad i = 1, \dots, m \\ & && g_j(\mathbf{x}) = v_j, \quad j = 1, \dots, M. \end{aligned}$$

Consider the optimal value of a GP p^* as a function of the perturbations \mathbf{u}, \mathbf{v} . The sensitivities of a GP with respect to the i th inequality constraint and l th equality constraint are defined as:

$$\begin{aligned} S_i &= \frac{\partial \log p^*(0, 0)}{\partial u_i} = \frac{\partial p^*(0, 0) / \partial u_i}{p^*(0, 0)}, \\ T_l &= \frac{\partial \log p^*(0, 0)}{\partial v_l} = \frac{\partial p^*(0, 0) / \partial v_l}{p^*(0, 0)}. \end{aligned}$$

A large sensitivity S_i with respect to an inequality constraint means that if the constraint is tightened (or loosened), the optimal value of GP increases (or decreases) considerably. Sensitivity can be obtained from the corresponding Lagrange dual variables of (2.6): $S_i = -\lambda_i$ and $T_j = -\nu_j$ where $\boldsymbol{\lambda}$ and $\boldsymbol{\nu}$ are the Lagrange multipliers of the inequality and equality constraints in (2.6), respectively.

3. Power Control by Geometric Programming: High SIR Case. GP in standard form can be used to formulate network resource allocation problems with nonlinear objectives under nonlinear QoS constraints. The key idea is that resources are often allocated proportional to some parameters, and when resource allocations are optimized over these parameters, we are maximizing an inverted posynomial subject to lower bounds on other inverted posynomials, which are equivalent to GP in standard form. This section presents how GP can be used to efficiently solve QoS constrained power control problems with nonlinear objectives, based on results in [14, 7].

Various schemes for power control, centralized or distributed, based on different transmission models and application needs, have been extensively studied since 1990s, *e.g.*, in [2, 13, 17, 20, 21, 22] and many other publications. This chapter summarizes the approach of formulating power control problems through GP (and an extension of GP called Signomial Programming). The key advantage is that globally optimal power allocations can be efficiently computed for a variety of nonlinear system-wide objectives and user QoS constraints, even when these nonlinear problems appear to be nonconvex optimization.

Power control problems occur in both cellular and multihop networks. The cellular power control problems can be viewed as special cases of the multihop problems where there is only one hop for all end-to-end transmissions. Since the transmission environment can be different along each link in multihop networks, power control schemes must consider each link along a flow's path.

3.1. Multihop wireless networks. Consider a wireless multihop network with n logical transmitter/receiver pairs. Transmit powers are denoted as P_1, \dots, P_n . Under Rayleigh fading, the power received from transmitter j at receiver i is given by $G_{ij} F_{ij} P_j$ where $G_{ij} \geq 0$ represents the path gain and is often modeled as proportional to $d_{ij}^{-\gamma}$ where d_{ij} is distance and γ is the power fall-off factor. We also let G_{ij} encompass antenna gain and coding gain. The Rayleigh fading between transmitter j and receiver i is given by F_{ij} , which are assumed to be independent and have unit mean. The distribution of the received power from transmitter j at receiver i is exponential

with mean value $\mathbf{E}[G_{ij}F_{ij}P_j] = G_{ij}P_j$. The SIR for the receiver on logical link i is:

$$(3.1) \quad \text{SIR}_i = \frac{P_i G_{ii} F_{ii}}{\sum_{j \neq i}^N P_j G_{ij} F_{ij} + n_i}$$

where n_i is the noise for receiver i .

The constellation size M used by a link can be closely approximated for MQAM modulations as follows: $M = 1 + \frac{-\phi_1}{\ln(\phi_2 \text{BER})} \text{SIR}$ where BER is the bit error rate and ϕ_1, ϕ_2 are constants that depend on the modulation type. Defining $K = \frac{-\phi_1}{\ln(\phi_2 \text{BER})}$ leads to an expression of the data rate R_i on the i th link as a function of SIR: $R_i = \frac{1}{T} \log_2(1 + K \text{SIR}_i)$, which will be approximated as

$$(3.2) \quad R_i = \frac{1}{T} \log_2(K \text{SIR}_i)$$

when $K \text{SIR}$ is much larger than 1. This approximation is reasonable either when the signal level is much higher than the interference level or when the spreading gain is large. For notational simplicity in the rest of this chapter, we redefine G_{ii} as K times the original G_{ii} , thus absorbing constant K into the definition of SIR.

The aggregate data rate for the system can then be written as the sum

$$R_{system} = \sum_i R_i = \frac{1}{T} \log_2 \left[\prod_i \text{SIR}_i \right].$$

So in the high SIR regime, aggregate data rate maximization is equivalent to maximizing a product of SIR. The system throughput is the aggregate data rate supportable by the system given a set of users with specified QoS requirements.

Outage probability is an important QoS parameter for reliable communication in wireless networks. A channel outage is declared and packets lost when the received SIR falls below a given threshold SIR_{th} , often computed from the BER requirement. Most systems are interference dominated and the thermal noise is relatively small, thus the i th link outage probability is

$$\begin{aligned} P_{o,i} &= \mathbf{Prob}\{\text{SIR}_i \leq \text{SIR}_{th}\} \\ &= \mathbf{Prob}\{G_{ii}F_{ii}P_i \leq \text{SIR}_{th} \sum_{k \neq i} G_{ik}F_{ik}P_k\}. \end{aligned}$$

The outage probability can be expressed as $P_{o,i} = 1 - \prod_{k \neq i} \frac{1}{1 + \frac{\text{SIR}_{th} G_{ik} P_k}{G_{ii} P_i}}$ [15],

which means that an upper bound on $P_{o,i} \leq P_{o,i,max}$ can be written as an upper bound on a posynomial in \mathbf{P} :

$$(3.3) \quad \prod_{k \neq i} 1 + \frac{\text{SIR}_{th} G_{ik} P_k}{G_{ii} P_i} \leq \frac{1}{1 - P_{o,i,max}}.$$

In wireless multihop networks with Rayleigh fading, we can use GP to efficiently maximize system throughput under user throughput constraints and outage probability constraints.

PROPOSITION 1. *The following nonlinear problem of optimizing power for constrained maximization of system throughput is a GP:*

$$(3.4) \quad \begin{aligned} &\text{maximize} && R_{system}(\mathbf{P}) \\ &\text{subject to} && R_i(\mathbf{P}) \geq R_{i,min}, \quad \forall i, \\ & && P_{o,i}(\mathbf{P}) \leq P_{o,i,max}, \quad \forall i, \\ & && P_i \leq P_{i,max}, \quad \forall i \end{aligned}$$

where the optimization variables are the transmit powers \mathbf{P} .

The objective is to maximize the system throughput, which is equivalent to minimizing the posynomial $\prod_i \text{ISR}_i$, where ISR_i is $\frac{1}{\text{SIR}}$. Each ISR_i is a posynomial in \mathbf{P} and the product of posynomials is again a posynomial. The first constraint is from the data rate demand $R_{i,\min}$ by each user. The second constraint represents the outage probability upper bounds $P_{o,i,\max}$ demanded by users using single links. These inequality constraints put upper bounds on posynomials of \mathbf{P} , as can be readily verified through (3.2,3.3). The fourth constraint is regulatory or system limitations on transmit powers. Thus (3.4) is indeed a GP, and efficiently solvable for global optimality.

There are several obvious variations of problem (3.4) that can be solved by GP, e.g., we can lower bound R_{system} as a constraint and maximize R_{i^*} for a particular user i^* , or maximize $\min_i R_i$ for maxmin fairness.

The objective function to be maximized can also be generalized to a weighted sum of data rates: $\sum_i w_i R_i$ where $\mathbf{w} \succeq 0$ is a given weight vector. This is still a GP because maximizing $\sum_i w_i \log \text{SIR}_i$ is equivalent to maximizing $\log \prod_i \text{SIR}_i^{w_i}$, which is in turn equivalent to minimizing $\prod_i \text{ISR}_i^{w_i}$. Now use auxiliary variables t_i , and minimize $\prod_i t_i^{w_i}$ over the original constraints in (3.4) plus the additional constraints $\text{ISR}_i \leq t_i$ for all i . This is readily verified to be a GP, and is equivalent to the original optimization.

In the general case of maximizing end-user utilities subject to per-link total throughput constraints that depend on power control policy, GP can also be used as a module in jointly optimal rate-power allocation [6].

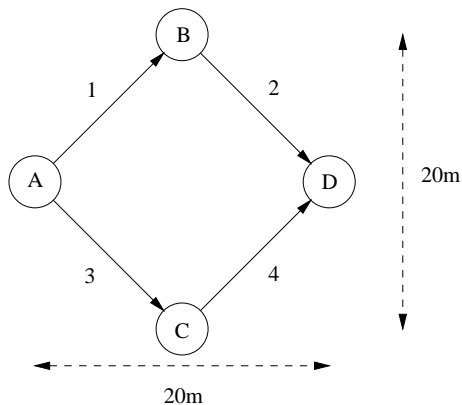


FIG. 3.1. A small wireless multihop network (Example 1).

Example 1. A simple four node multihop network, shown in Figure 3.1, is considered in the following numerical example. There are two connections $A \rightarrow B \rightarrow D$ and $A \rightarrow C \rightarrow D$. Nodes A and D , as well as B and C , are separated by a distance of 20m. Path gain between a transmitter and a receiver is the distance to the power -4 . Each link has a maximum transmit power of 1mW. All nodes use MQAM modulation. The minimum data rate for each connection is 100bps, and the target BER is 10^{-3} . Assuming Rayleigh fading, we require outage probability be smaller than 0.1 on all links for an SIR threshold of 10dB. Spreading gain is 200. Using GP formulation (3.4), we find the maximized system throughput $R^* = 216.8\text{kbps}$, $R_i^* = 54.2\text{kbps}$ for each link, $P_1^* = P_3^* = 0.709\text{mW}$ and $P_2^* = P_4^* = 1\text{mW}$. The

resulting optimized SIR is 21.7dB on each link.

For this topology, we also consider an illustrative example of admission control and pricing. Three new users U_1 , U_2 , and U_3 are going to arrive to the network in order. U_1 and U_2 require 30kbps sent along the upper path $A \rightarrow B \rightarrow D$, while U_3 requires 10kbps sent from $A \rightarrow B$. All three users require the outage probability to be less than 0.1. When U_1 arrives at the system, her price is the baseline price. Next, U_2 arrives, and her QoS demands decrease the maximum system throughput from 216.82kbps to 116.63kbps, so her price is the baseline price plus an amount proportional to the reduction in system throughput. Finally, U_3 arrives, and her QoS demands produce no feasible power allocation solution, so she is not admitted to the system.

We now turn to delay and buffer overflow properties to be included in constraints or objective function of power control optimization. The average delay a packet experiences traversing a network is another important design consideration in many applications. Queuing delay is often the primary source of delay, particularly for bursty data traffic in multihop networks. A node i first buffers the received packets in a queue and then transmits these packets at a rate R set by the SIR on the egress link, which is in turn determined by the transmit powers \mathbf{P} . A FIFO queuing discipline is used here for simplicity. The approach can be extended to other disciplines. Routing is assumed to be fixed or only changes infrequently, and is feed-forward with all packets visiting a node at most once.

Packet traffic entering the multihop network at the transmitter of link i is assumed to be Poisson with parameter λ_i and to have an exponentially distributed length with parameter Γ . Using the model of an $M/M/1$ queue, the probability of transmitter i having a backlog of $N_i = k$ packets to transmit is well known to be $\mathbf{Prob}\{N_i = k\} = (1 - \rho)\rho^k$ where $\rho = \frac{\lambda_i}{\Gamma R_i(\mathbf{P})}$, and the expected delay is $\frac{1}{\Gamma R_i(\mathbf{P}) - \lambda_i}$. Under the feed-forward routing and Poisson input assumptions, Burke's theorem can be applied. Thus the total packet arrival rate at node i is $\Lambda_i = \sum_{j \in I(i)} \lambda_j$ where $I(i)$ is the set of connections traversing node i . The expected delay \bar{D}_i can be written as

$$(3.5) \quad \bar{D}_i = \frac{1}{\Gamma R_i(\mathbf{P}) - \Lambda_i}.$$

A bound $\bar{D}_{i,max}$ on \bar{D}_i can thus be written as $\frac{1}{\frac{T}{\Gamma} \log_2(\text{SIR}_i) - \Lambda_i} \leq \bar{D}_{i,max}$, or equivalently, $\text{ISR}_i(\mathbf{P}) \leq 2^{-\frac{T}{\Gamma}(\bar{D}_{i,max}^{-1} + \Lambda_i)}$, which is an upper bound on a posynomial ISR of \mathbf{P} .

PROPOSITION 2. *The following nonlinear problem of optimizing powers to maximize system throughput, subject to constraints on outage probability and expected delay, is a GP:*

$$(3.6) \quad \begin{aligned} & \text{maximize} && R_{\text{system}}(\mathbf{P}) \\ & \text{subject to} && \bar{D}_i(\mathbf{P}) \leq \bar{D}_{i,max}, \quad \forall i, \\ & && \text{Same constraints as in problem (3.4)} \end{aligned}$$

where the optimization variables are the transmit powers \mathbf{P} .

The probability P_{BO} of dropping a packet due to buffer overflow at a node is also important in several applications. It is again a function of \mathbf{P} and can be written as $P_{BO,i} = \mathbf{Prob}\{N_i > B\} = \rho^{B+1}$ where B is the buffer size and $\rho = \frac{\Lambda_i}{\Gamma R_i(\mathbf{P})}$. Setting an upper bound $P_{BO,i,max}$ on the buffer overflow probability also gives a posynomial lower bound constraint in \mathbf{P} : $\text{ISR}_i(\mathbf{P}) \leq 2^{-\Psi}$ where $\Psi = \frac{T\Lambda_i}{\Gamma(P_{BO,i,max})^{\frac{1}{B+1}}}$.

PROPOSITION 3. *The following nonlinear problem of optimizing powers to maximize system throughput, subject to constraints on outage probability, expected delay, and the probability of buffer overflow, is a GP:*

$$(3.7) \quad \begin{aligned} & \text{maximize} && R_{\text{system}}(\mathbf{P}) \\ & \text{subject to} && P_{BO,i}(\mathbf{P}) \leq P_{BO,i,\text{max}}, \quad \forall i, \\ & && \text{Same constraints as in problem (3.6)} \end{aligned}$$

where the optimization variables are the transmit powers \mathbf{P} .

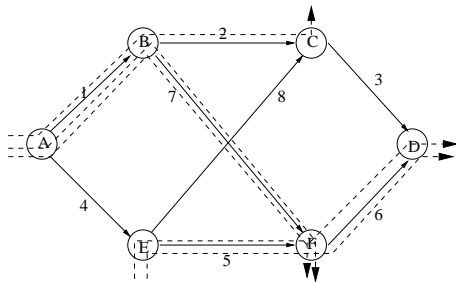


FIG. 3.2. *Topology and flows in a multihop wireless network (Example 2).*

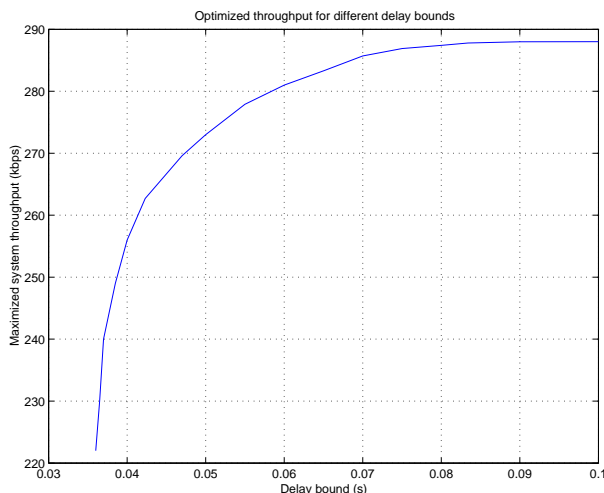


FIG. 3.3. *Optimal tradeoff between maximized system throughput and average delay constraint (Example 2).*

Example 2. Consider a numerical example of the optimal tradeoff between maximizing the system throughput and bounding the expected delay for the network shown in Figure 3.2. There are six nodes, eight links, and five multihop connections. All sources are Poisson with intensity $\lambda_i = 200$ packets per second, and exponentially distributed packet lengths with an expectation of 100 bits. The nodes use CDMA transmission scheme with a symbol rate of 10k symbols per second and the spreading gain is 200. Transmit powers are limited to 1mW and the target BER is 10^{-3} . The path loss matrix is calculated based on a power falloff of d^{-4} with the distance d , and a separation of 10m between any adjacent nodes along the perimeter of the network.

Figure 3.3 shows the maximized system throughput for different upper bound numerical values in the expected delay constraints, obtained by solving a sequence of GPs, one for each point on the curve. There is no feasible power allocation to achieve delay smaller than 0.036s. As the delay bound is relaxed, the maximized system throughput increases sharply first, then more slowly until the delay constraints are no longer active. Comparing performance with several existing power control algorithms, which cannot handle the nonlinear objectives like that of (3.4), we find that either the delay bound is violated or the resulted throughput is not maximized by the existing algorithms. In contrast, GP efficiently returns the globally optimal tradeoff between system throughput and queuing delay.

Obviously, the GP method in this subsection can also efficiently compute the globally optimal power control if the objective is to minimize \bar{D}_i or $P_{BO,i}$, subject to the constraints of lower bounds on system or individual throughput, and upper bounds on per-link or per-path outage probability. In addition to efficient computation of the globally optimal power allocation with nonlinear objectives and constraints, GP can also be used for admission control based on feasibility study described in subsection 2.2 and for determining which QoS constraint is a performance bottleneck, *i.e.*, meet tightly at the optimal power allocation, based on sensitivity analysis in subsection 2.2.

3.2. Cellular wireless networks. GP-based power control also applies to cellular wireless networks with one-hop transmission from mobile users to base stations, extending the scope of power control problems solvable by the classical solution in CDMA systems that equalizes SIRs, and those by the iterative algorithms (*e.g.*, in [2, 13, 17]) that minimize total power subject to SIR constraints. For example, consider a single cell with one base station and N mobile users indexed by k .

PROPOSITION 4. *The following problem of maximizing the SIR of a particular user i^* , subject to a variety of constraints for all users, is a GP:*

$$\begin{aligned} & \text{maximize} && \text{SIR}_{i^*} \\ & \text{subject to} && \text{SIR}_k \geq \text{SIR}_{k,\min}, \quad \forall k, \\ & && \sum_{j \in I_k} P_j d_j^{-\gamma_j} \alpha_j < c_k, \quad \forall k, \\ & && P_{k1} G_{k1} = P_{k2} G_{k2}, \\ & && P_k \leq P_{k,\max}, \quad \forall k, \\ & && P_k \geq 0, \quad \forall k. \end{aligned}$$

The first constraint, equivalent to $R_k \geq R_{k,\min}$, sets a floor on the SIR of other users and protects these users from user i^* increasing her transmit power excessively. The second constraint sets a bound on the level of interference other users in some index set I_k can create for user k . The third constraint reflects the classical power control criterion in solving the near-far problem in CDMA systems: the expected received power from one transmitter $k1$ must equal that from another $k2$. All constraints are verified to be inequality upper bounds on posynomials.

Alternatively, we can use GP to maximize a weighted sum of data rates from all users, or the minimum SIR among all users. The maxmin fairness objective:

$$\text{maximize} \quad \min_k \text{SIR}_k$$

can be accommodated in GP based power control because it can be turned into equivalently maximizing an auxiliary variable t such that $\text{SIR}_k \geq t$, $\forall k$, which are polynomial objective and constraints in (\mathbf{P}, t) .

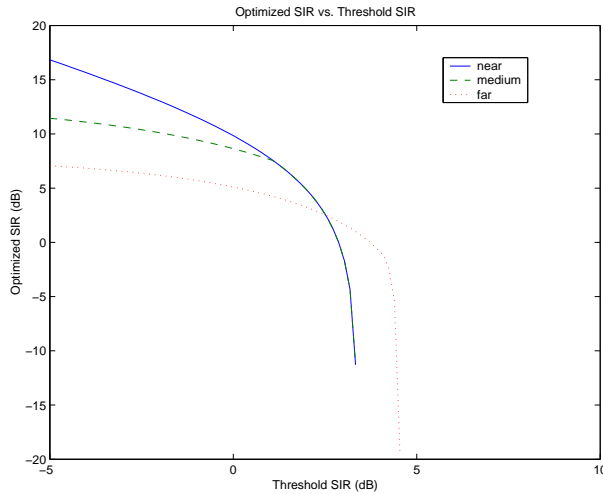


FIG. 3.4. Results of constrained optimization of power control in a cellular network, with implications for admission control and pricing scheme (Example 3).

Example 3. A simple system comprised of five users is used for a numerical example. First, the five users are spaced at distances d of 1, 5, 10, 15, and 20 units from the base station. The power drop off factor $\gamma = 4$. Each user has a maximum power constraint of $P_{max} = 0.5mW$. The noise power is $0.5\mu W$ for all users. The SIR of all users, other than the user we are optimizing for, must be greater than a common threshold SIR level β , which is varied to observe the effect on the optimized user's SIR. This is done independently for the near user at $d = 1$, a medium distance user at $d = 15$, and the far user at $d = 20$. The results are plotted in Figure 3.4.

Several interesting effects are illustrated. First, when the required threshold SIR for the non-optimized users is high, there are no feasible power control solutions. At moderate threshold SIR, as β is decreased, the optimized SIR initially increases rapidly. This is because it is allowed to increase its own power by the sum of the power decrease in the four other users, and the noise is relatively insignificant. At low threshold SIR, the noise becomes more significant and the power trade-off from the other users less significant, so the curve starts to bend over. Eventually, the optimized user reaches its upper bound on power and cannot utilize the excess power allowed by the lower threshold SIR for other users. Therefore, during this stage, the only gain in the optimized SIR is the lower power transmitted by the other users. This is exhibited by the transition from a sharp bend in the curve to a much shallower sloped curve. We also note that the most distant user in the constraint set dictates feasibility.

Example 4. A large scale simulation is conducted for 100 users scattered in a geographical area as shown in Figure 3.5. The optimized SIR for user 1 is plotted against the SIR threshold for the other 99 users in Figure 3.6.

4. Power Control by Geometric Programming: Medium to Low SIR Case. There are two main limitations in the GP-based power control methods discussed so far, and both can be overcome as briefly discussed in this section and the appendix, respectively.

The first limitation is the assumption that SIR is much larger than 0dB, which can be removed by condensation method for Signomial Programming (SP). When

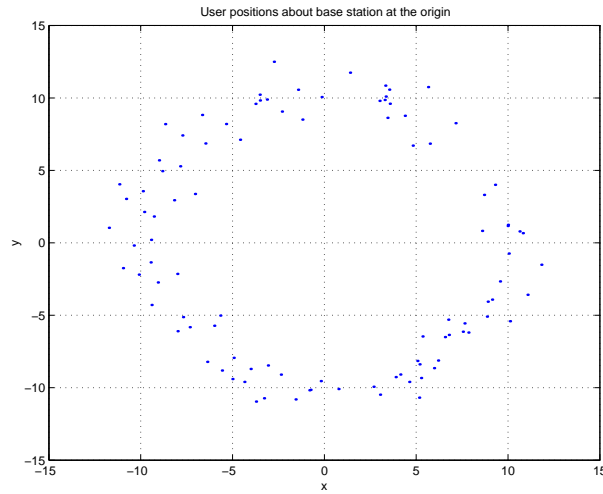


FIG. 3.5. A large cellular wireless network with 100 users (Example 4).

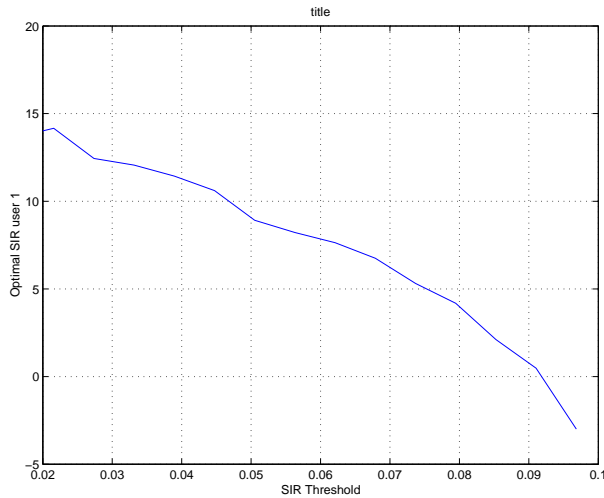


FIG. 3.6. A large scale numerical example for constrained SIR optimization with 100 users (Example 4).

SIR is not much larger than 0dB, the approximation of $\log(1 + \text{SIR})$ as $\log \text{SIR}$ does not hold. Unlike SIR, which is an inverted posynomial, $1 + \text{SIR}$ is not an inverted posynomial. Instead, $\frac{1}{1+\text{SIR}}$ is a ratio between two posynomials:

$$\frac{\sum_{j \neq i} G_{ij} P_j + n_i}{\sum_j G_{ij} P_j + n_i}.$$

4.1. Signomial programming. To overcome this issue, GP can be extended to SP: minimizing a signomial subject to upper bound inequality constraints on signomials, where a signomial is a sum of monomials, possibly with *negative* multiplicative

coefficients:

$$s(\mathbf{x}) = \sum_{i=1}^N c_i g_i(\mathbf{x})$$

where $\mathbf{c} \in \mathbf{R}^N$ and $g_i(\mathbf{x})$ are monomials.

We first convert a SP into a Complementary GP, which allows upper bound constraints on the ratio between two posynomials, and then apply a monomial approximation iteratively. This is called the condensation method [3, 11], which is an instance of the cutting-plane method for nonlinear programming.

The conversion from a SP into a Complementary GP is trivial. An inequality in SP of the following form

$$f_{i1}(\mathbf{x}) - f_{i2}(\mathbf{x}) \leq 1,$$

where f_{i1}, f_{i2} are posynomials, is clearly equivalent to

$$\frac{f_{i1}(\mathbf{x})}{1 + f_{i2}(\mathbf{x})} \leq 1.$$

Now we have two choices to make the monomial approximation. One is to approximate the denominator $1 + f_{i2}(\mathbf{x})$ with a monomial but leave the numerator $f_{i1}(\mathbf{x})$ as a posynomial. This is called the (single) condensation method, and results in a GP approximation of a SP. An iterative procedure can again be carried out: given a feasible \mathbf{x}^k , from which a monomial approximations using $\boldsymbol{\alpha}(\mathbf{x}^k)$ can be made and a GP formed, from which an optimizer can be computed and used as \mathbf{x}^{k+1} , which becomes the starting point for the next iteration. This sequence of computation of \mathbf{x} may converge to \mathbf{x}^* , an optimizer of the original SP.

There are many ways to make a monomial approximation of a posynomial. One possibility is based on the following simple inequality: arithmetic mean is greater than or equal to geometric mean, *i.e.*,

$$\sum_i \alpha_i v_i \geq \prod_i v_i^{\alpha_i},$$

where $\mathbf{v} \succ 0$ and $\boldsymbol{\alpha} \succeq 0$, $\mathbf{1}^T \boldsymbol{\alpha} = 1$. Letting $u_i = \alpha_i v_i$, we can write this basic inequality as

$$\sum_i u_i \geq \prod_i \left(\frac{u_i}{\alpha_i} \right)^{\alpha_i}.$$

Let $\{u_i(\mathbf{x})\}$ be the monomial terms in a posynomial $f(\mathbf{x}) = \sum_i u_i(\mathbf{x})$. A lower bound inequality on posynomial $f(\mathbf{x})$ can now be approximated by an upper bound inequality on the following monomial:

$$(4.1) \quad \prod_i \left(\frac{u_i(\mathbf{x})}{\alpha_i} \right)^{-\alpha_i}.$$

This approximation is in the conservative direction because the original constraint is now tightened. There are many choices of $\boldsymbol{\alpha}$. One possibility is to let

$$\alpha_i(\mathbf{x}) = u_i(\mathbf{x})/f(\mathbf{x}), \quad \forall i,$$

which obviously satisfies the condition that $\alpha \succ 0$ and $\mathbf{1}^T \alpha = 1$. Given an α for each lower bound posynomial inequality, a standard form GP can be obtained based on the above geometric mean approximation.

Another choice is to make the monomial approximation for both the denominator $1 + f_{i2}(\mathbf{x})$ and numerator posynomials $f_{i2}(\mathbf{x})$. That turns all the constraints into monomials, and after a log transformation, approximate SP as a linear program. This is called the double condensation method, and a similar iterative procedure can be carried out as in the last paragraph. A key difference from the condensation method is that this LP approximation always generate solutions that are infeasible in the original SP. Therefore in the k th step of the iteration, the most violated constraint is condensed at \mathbf{x}^k , *i.e.*, the monomial approximation is applied to this constraint inequality using $\alpha(\mathbf{x}^k)$. The resulting new constraint is added to the LP approximation for the $(k+1)$ th step of the iteration. The solution \mathbf{x}^* at which all constraints in the original SP are satisfied is an optimum of the SP.

4.2. Applications to power control. GP-based power control problems in medium to small SIR regimes become signomial programs, which can be solved by single or double condensation method. We focus on single condensation method here. Consider a representative problem formulation of maximizing total system throughput in a cellular wireless network subject to user rate and outage probability constraints:

$$\begin{aligned} & \text{maximize} && R_{system}(\mathbf{P}) \\ & \text{subject to} && R_i(\mathbf{P}) \geq R_{i,min}, \quad \forall i, \\ & && P_{o,i}(\mathbf{P}) \leq P_{o,i,max}, \quad \forall i, \\ & && P_i \leq P_{i,max}, \quad \forall i, \end{aligned}$$

which is explicitly written out as:

$$(4.2) \quad \begin{aligned} & \text{minimize} && \prod_{i=1}^N \frac{1}{1+SIR_i} \\ & \text{subject to} && (2^{TR_{i,min}} - 1) \frac{1}{SIR_i} \leq 1, \quad i = 1, \dots, N, \\ & && (SIR_{th})^{N-1} (1 - P_{o,i,max}) \prod_{i \neq k}^N \frac{G_{ik} P_{ik}}{G_{ii} P_i} \leq 1, \quad i = 1, \dots, N, \\ & && P_i (P_{i,max})^{-1} \leq 1, \quad i = 1, \dots, N. \end{aligned}$$

All the constraints are posynomials. However, the objective is not a posynomial, but a ratio between two posynomials. This power control problem is a SP (equivalently, a Complementary GP), and can be solved by condensation method by solving a series of GPs. Specifically, we have the following algorithm:

STEP 0: Choose an initial feasible \mathbf{P} .

STEP 1: Evaluate the denominator posynomial of the (4.2) objective function with the given \mathbf{P} .

STEP 2: Compute for each term i in this posynomial,

$$\delta_i = \frac{\text{value of } i\text{-th term in posynomial}}{\text{value of posynomial from STEP 2}}.$$

STEP 3: Condense the denominator posynomial of the (4.2) objective function into a monomial using (4.1) with weights δ_i .

STEP 4: Solve the resulting GP using interior point method.

STEP 5: Go to STEP 1 using \mathbf{P} of STEP 4.

STEP 6: Terminate the k -th loop if $\|\mathbf{P}^{(k)} - \mathbf{P}^{(k-1)}\| \leq \epsilon$ where ϵ is the error tolerance for exit condition.

As condensing the objective in the above problem gives us an underestimate of the objective value, each GP in the condensation iteration loop tries to improve the accuracy of the approximation to a particular minimum in the original feasible region.

Example 5. We consider a cellular wireless network with 3 users. Let $T = 10^{-6}$ s, $G_{ii} = 1.5$, and generate $G_{ij}, i \neq j$, as independent random variables uniformly distributed between 0 and 0.3. Threshold SIR is $SIR_{th} = -10$ dB, and minimal data rate requirements are 100 kbps, 600 kbps and 1000 kbps for logical links 1, 2 and 3 respectively. Maximal outage probabilities are 0.01 for all links, and maximal transmit powers are 3mW, 4mW and 5mW for link 1, 2 and 3 respectively.

For each instance of SP power control (4.2), we pick a random initial feasible power vector \mathbf{P} uniformly between 0 and \mathbf{P}_{max} . Fig. 4.1 compares the maximized total network throughput achieved over five hundred sets of experiments with different initial vectors. With (single) condensation method, SP converges to different optima over the entire set of experiments, achieving (or coming very close to) the global optimum at 5290 bps (96% of the time) and a local optimum at 5060 bps (4% of the time), thus very likely to converge to or very close to the global optimum. Figure 4.2 shows the number of GP iterations required by condensation method over the same set of experiments, where the average number is 15 GPs if an extremely tight exit condition is picked for SP condensation iteration: $\epsilon = 1 \times 10^{-10}$. This average can be substantially reduced by using a larger ϵ .

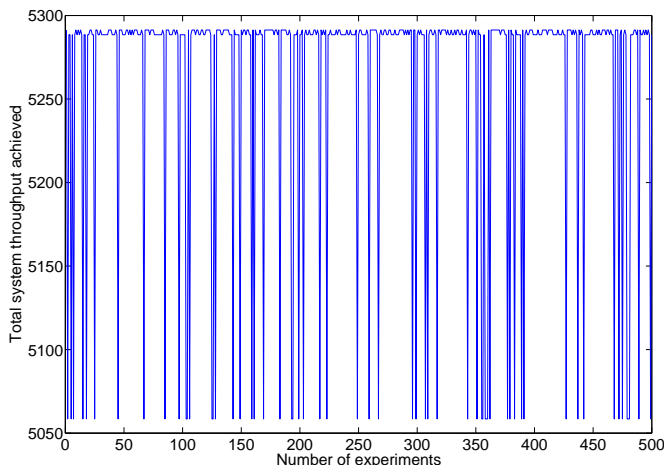


FIG. 4.1. *Maximized total system throughput achieved by (single) condensation method for 500 different initial feasible vectors (Example 5).*

Other constant parameter values are tried, and a particular instance of these further trials is shown in Table 4.1, where the maximized total system throughput and the corresponding power vector are shown for exhaustive search (which always generates the globally optimal results), condensation method (which solves the SP formulation through a series of GPs), and high SIR approximation (which approximates $\log(1 + SIR)$ as $\log(SIR)$, thus turning SP formulation into a single GP). Similar comparisons are shown for fifty trials in Figure 4.3. It is observed that condensation method attains a maximized total system throughput that comes extremely close to the globally optimum computed by exhaustive search for these nonlinear power con-

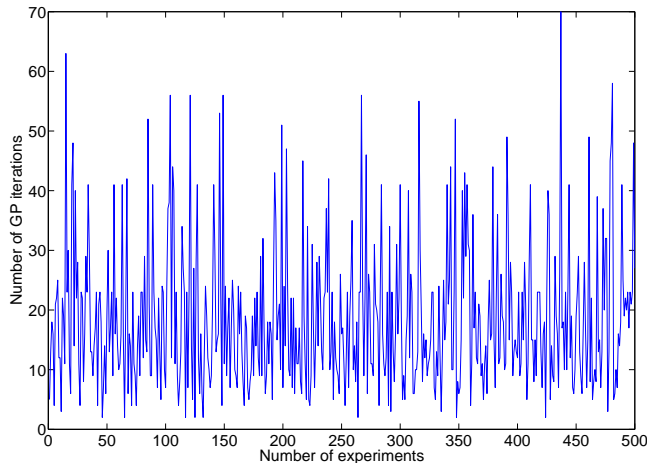


FIG. 4.2. The number of GP iterations required by (single) condensation method for 500 different initial feasible vectors (Example 5).

trol problems that cannot be turned into convex optimization. For some of these trials, the parameters are such that the high SIR approximation turn out to also work well.

Method	Maximized total system throughput (kbps)	P_1^*	P_2^*	P_3^*
Exhaustive search	4286	2.0	3.0	4.0
Single condensation	4286 by solving 3 GPs	2.0	3.0	4.0
High SIR approximation	4246 by solving 1 GP	2.0	2.94	2.94

TABLE 4.1
A particular instance of Example 5.

Example 6. In another set of numerical experiments summarized in Figure 4.4, high SIR approximation method performs substantially worse than condensation method. By attempting to solve SP through a series of GPs, condensation method sometimes computes the globally optimal power allocation as verified by exhaustive search, and comes close to the global optimum at other instances.

Example 7. A slightly larger cellular system with 6 users is then studied. The number of users is still small enough for exhaustive search to be conducted and to establish the benchmark of globally optimal power control. Performance and computational complexity of SP condensation method for 300 different initial feasible power allocations are shown in Figures 4.5 and 4.6, respectively. With a few exceptions, SP condensation returns the globally optimal power allocation. By using a more relaxed but sufficiently accurate exit condition $\epsilon = 10^{-3}$, the average number of GP iterations needed is reduced to 11 even though the problem size doubles compared to Example 5.

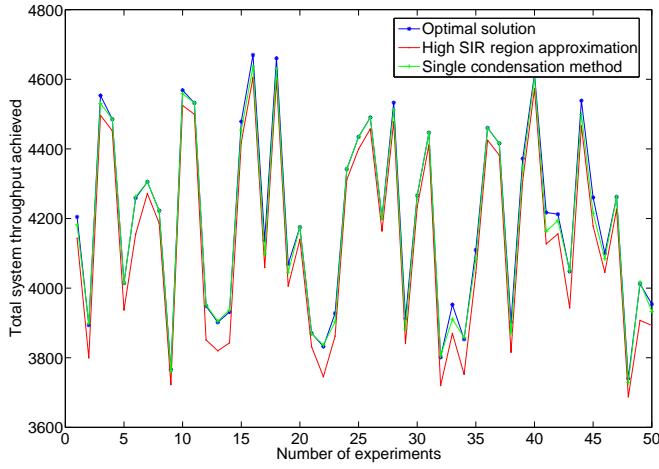


FIG. 4.3. A comparison of the total system throughput achieved by high SIR approximation and single condensation method with the optimal solution by exhaustive search (Example 5).

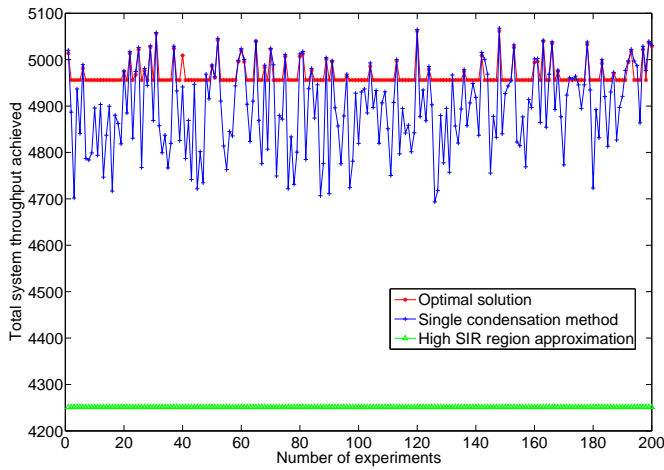


FIG. 4.4. A comparison of the total system throughput achieved by high SIR approximation and single condensation method with the optimal solution by exhaustive search(Example 6).

4.3. Extensions. The optimum of power control produced by condensation method may be a local one. The following heuristics of solving a series of SPs (each solved through a series of GPs) can be further applied to help find the global optimum. After the original SP (4.2) is solved, a slightly modified SP is formulated and

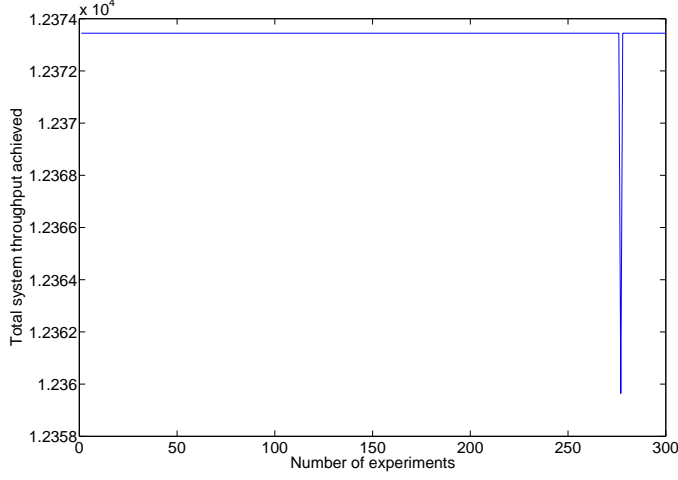


FIG. 4.5. Maximized total system throughput achieved by (single) condensation method for 300 different initial feasible vectors (Example 7).

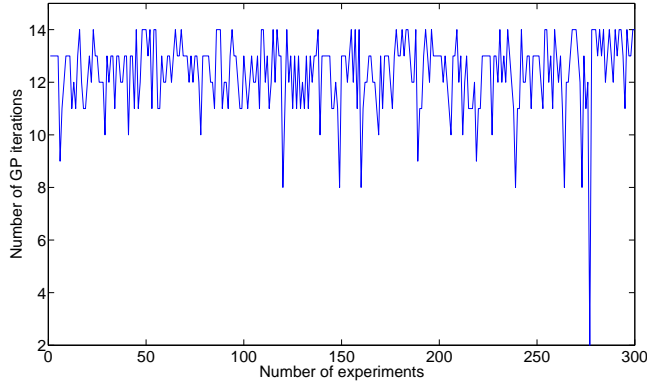


FIG. 4.6. The number of GP iterations required by (single) condensation method for 300 different initial feasible vectors (Example 7).

solved:

$$\begin{aligned}
 & \text{minimize} && t \\
 & \text{subject to} && \prod_{i=1}^N \frac{1}{1+\text{SIR}_i} \leq t, \\
 & && t \leq \frac{t_0}{\alpha}, \\
 (4.3) \quad & && (2^{\overline{R}_{min,i}} - 1) \frac{1}{\text{SIR}_i} \leq 1, \quad i = 1, \dots, N, \\
 & && (\text{SIR}_{th})^{N-1} (1 - P_{o,i,max}) \prod_{i=1, i \neq k}^N \frac{G_{ik} P_{ik}}{G_{ii} P_i} \leq 1, \quad i = 1, \dots, N, \\
 & && P_i (P_i^{max})^{-1} \leq 1, \quad i = 1, \dots, N,
 \end{aligned}$$

where α is a constant slightly larger than 1. At each iteration of a modified SP, the previous computed optimum value is set to constant t_0 and the modified problem (4.3) is solved to yield an objective value that is better than the objective value of the previous SP by at least α . The auxiliary variable t is introduced so as to turn the

problem formulation into SP in (\mathbf{P}, t) .

Example 8. The above heuristics is applied to the instances of Example 5 where solving SP returns a locally optimal power allocation, and is found to obtain the globally optimal solution within 1 or 2 rounds of solving additional SPs (4.3).

We have discussed a power control problem (4.2) where the objective function needs to be condensed. The method is also applicable if some constraint functions are signomials and need to be condensed. For example, consider the case of differentiated services where a user expects to obtain a predicted QoS relatively better than the other users. We may have a proportional delay differentiation model where a user who pays more tariff obtains a delay proportionally lower as compared to users who pay less. Then for a particular ratio between any user i and j , σ_{ij} , we have

$$(4.4) \quad \frac{\bar{D}_i}{\bar{D}_j} = \sigma_{ij},$$

which, by (3.5), is equivalent to

$$(4.5) \quad \frac{1 + \text{SIR}_j}{(1 + \text{SIR}_i)^{\sigma_{ij}}} = 2^{(\lambda_j - \sigma_{ij}\lambda_i)T/\Gamma}.$$

The denominator on the left hand side is a posynomial raised to a positive power. Therefore, double condensation method can be readily used to solve the proportional delay differentiation problem because the function on the left hand side can be condensed to a monomial, and a monomial equality constraint is allowed in GP.

5. Conclusions. Power control problems with nonlinear objective and constraints may seem to be difficult to solve for global optimality. However, when SIR is much larger than 0dB, GP can be used to turn these problems, with a variety of possible combinations of objective and constraint functions involving data rate, delay, and outage probability, into intrinsically tractable convex formulations. Then interior point algorithms can efficiently compute the globally optimal power allocation even for a large network. Feasibility and sensitivity analysis of GP naturally lead to admission control and pricing schemes. When the high SIR approximation cannot be made, these power control problems become SP and may be solved by the heuristics of condensation method. As outlined in the appendix, distributed solutions for GP-based power control in ad hoc networks present a promising next step of research.

Acknowledgement. We would like to acknowledge the collaborations on GP based resource allocation with Stephen Boyd at Stanford University, Daniel Palomar at Princeton University, and Arak Sutivong at Qualcomm.

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Appendix: Distributed implementation. The second limitation for GP based power control is the need for centralized computation if a GP is solved by interior point methods. The GP formulations of power control problems can also be solved by the general method of distributed algorithm for GP in [19, 7], which is summarized through an illustrative example in this appendix. Furthermore, the special structure of coupling, *i.e.*, all coupling among the logical links are due to the total interference term, which is a linear function of \mathbf{P} , can be used to further reduce the amount of message passing among the transmitters [19].

Consider the following standard form GP with one posynomial inequality constraint (the general case of multiple posynomial inequality constraints is a simple extension):

$$\begin{aligned} & \text{minimize} && \sum_i c_i h_i(\mathbf{x}) \\ & \text{subject to} && \sum_i d_i g_i(\mathbf{x}) \leq 1 \end{aligned}$$

where h_i and g_i are monomials, and $\mathbf{c}, \mathbf{d} \succeq 0$ are constants. The key idea is to utilize the additive structure of posynomials in a Lagrangian dual decomposition, and then use a distributed gradient method to solve the dual problem.

First, we need to decompose the coupling of the variables across the monomial terms in the objective and constraint functions. Consider an example GP in \mathbf{x} :

$$\begin{aligned} & \text{minimize} && x_1 x_3 + x_2 x_3 + x_3 \\ & \text{subject to} && x_1 x_2 + x_2 x_3 \leq 1, \end{aligned}$$

which is equivalent to the following GP in $(\mathbf{x}, \mathbf{y}, \mathbf{z})$:

$$\begin{aligned} & \text{minimize} && x_1 y_{13} + x_2 y_{23} + x_3 \\ & \text{subject to} && x_1 z_{12} + x_2 z_{23} \leq 1, \\ & && y_{13} = x_3, \\ & && y_{23} = x_3, \\ & && z_{12} = x_2, \\ & && z_{23} = x_3. \end{aligned}$$

Here \mathbf{y} are used to decouple the monomials in the objective function and \mathbf{z} to decouple those in the constraint function. The first subscript indexes the term in the posynomial while the second subscript indexes the variable being decoupled.

The Lagrangian can be formed:

$$\begin{aligned} L(\mathbf{x}, \mathbf{y}, \mathbf{z}, \boldsymbol{\gamma}, \boldsymbol{\lambda}, \theta) &= x_1 y_{13} + x_2 y_{23} + x_3 + \theta(x_1 z_{12} + x_2 z_{23} - 1) \\ &+ \gamma_{13}(y_{13} - x_3) + \gamma_{23}(y_{23} - x_3) + \lambda_{12}(z_{12} - x_2) + \lambda_{23}(z_{23} - x_3) \end{aligned}$$

where $\theta \geq 0$ is the dual variable for the posynomial inequality constraint, $\boldsymbol{\gamma}$ are the dual variables for the additional equality constraints due to decoupling of the objective function, and $\boldsymbol{\lambda}$ are the dual variables for the additional equality constraints due to decoupling of the constraint function.

The Lagrangian can be decoupled into four sets of terms, one for each user and another that only involves the dual variables:

$$L(\mathbf{x}, \mathbf{y}, \mathbf{z}, \boldsymbol{\gamma}, \boldsymbol{\lambda}, \theta) = L_1 + L_2 + L_3 - \theta$$

where

$$\begin{aligned} L_1 &= x_1 y_{13} + \theta x_1 z_{12} + \gamma_{13} y_{13} + \lambda_{12} z_{12} \\ L_2 &= x_2 y_{23} + \theta x_2 z_{23} + \gamma_{23} y_{23} + \lambda_{23} z_{23} - \lambda_{12} x_2 \\ L_3 &= x_3 - \gamma_{13} x_3 - \gamma_{23} x_3 - \lambda_{23} x_3. \end{aligned}$$

Each term $L_i, i = 1, 2, 3$, can be optimized over only the primal variables $\mathbf{x}, \mathbf{y}, \mathbf{z}$ whose first subscripts are i , *i.e.*, only the local variables. This parallel local optimization computes the dual function $g(\boldsymbol{\gamma}, \boldsymbol{\lambda}, \theta) = L(\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*, \boldsymbol{\gamma}, \boldsymbol{\lambda}, \theta)$. Minimization of

the dual function can be carried out by an iterative gradient method:

$$\begin{aligned}\theta(t+1) &= \theta(t) + \alpha(t)(x_1 z_{12} + x_2 z_{23} - 1) \\ \gamma_{ij}(t+1) &= \gamma_{ij}(t) + \alpha(t)(y_{ij} - x_j) \\ \lambda_{ij}(t+1) &= \lambda_{ij}(t) + \alpha(t)(z_{ij} - x_j)\end{aligned}$$

where t is the iteration number and $\alpha(t)$ are step sizes, which in general can be different in each of the above update equations. Appropriate choices of $\alpha(t)$ ensures convergence of this iteration based on dual decomposition.

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