ELE539A: Optimization of Communication Systems
Lecture 6: Quadratic Programming, Geometric Programming, and Applications

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Lecture Outline

- Quadratic constrained quadratic programming (QCQP)
- Least-squares
- Second order cone programming (SOCP)
- Dual quadratic programming
- Geometric programming (GP)
- Dual geometric programming
- Wireless network power control

Thanks: Stephen Boyd (some materials and graphs from Boyd and Vandenberghe)

Convex QCQP

- (Convex) QP (with linear constraints) in x:

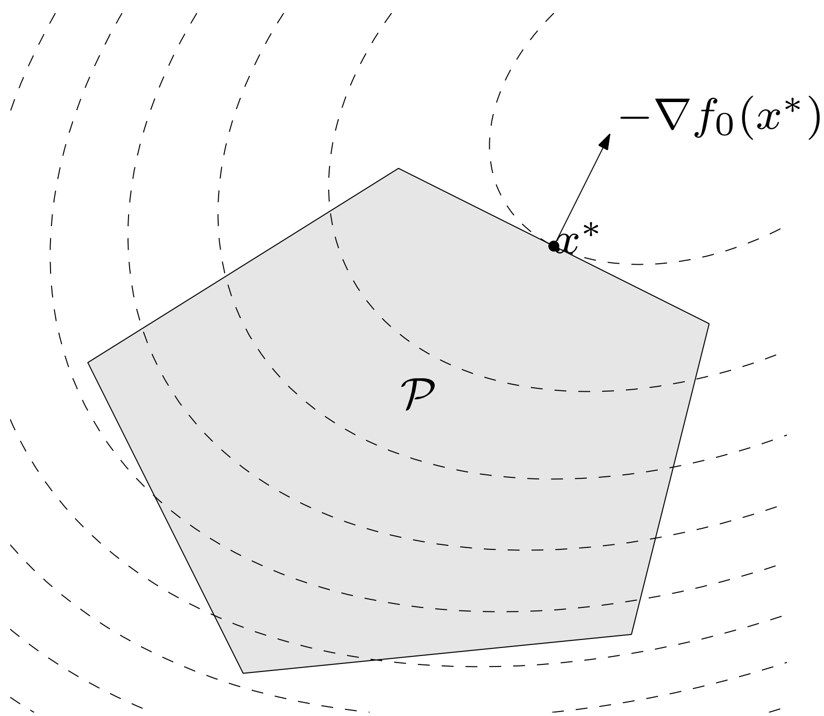
$$\begin{array}{ll} \text{minimize} & (1/2)x^T Px + q^T x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

where $P \in \mathbf{S}^n_+, G \in \mathbf{R}^{m\times n}, A \in \mathbf{R}^{p\times n}$

- (Convex) QCQP in x:

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, 2, \ldots, m \\ & Ax = b \end{array}$$

where $P \in \mathbf{S}^n_+, \quad i = 0, \ldots, m$


$-\nabla f_0(x^*)$
x^*
$\mathcal{P}$

Least-squares

- Minimize $\|Ax - b\|_2^2 = x^T A^T Ax - 2b^T Ax + b^T b$ over x. Unconstrained QP, Regression analysis, Least-squares approximation

Analytic solution: $x^* = A^\dagger b$ where, for $A \in \mathbf{R}^{m \times n}$, $A^\dagger = (A^T A)^{-1} A^T$ if rank of A is n, and $A^\dagger = A^T (AA^T)^{-1}$ if rank of A is m. If not full rank, then by singular value decomposition.

- Constrained least-squares (no general analytic solution). For example:

$$
\begin{array}{ll}
\text{minimize} & \|Ax - b\|_2^2 \\
\text{subject to} & l_i \leq x_i \leq u_i, \;\; i = 1, \ldots, n
\end{array}
$$

LP with Random Cost

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

Cost $c \in \mathbf{R}^n$ is random, with mean $\bar{c}$ and covariance Ω

Expected cost: $\bar{c}^T x$. Cost variance $x^T \Omega x$

Minimize both expected cost and cost variance (with a weight γ):

$$\begin{array}{ll} \text{minimize} & \bar{c}^T x + \gamma x^T \Omega x \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

SOCP

Second Order Cone Programming:

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m \\ & Fx = g \end{array}$$

Variables: $x \in \mathbf{R}^n$. And $A_i \in \mathbf{R}^{n_i \times n}, F \in \mathbf{R}^{p \times n}$

- If $c_i = 0$, $\forall i$, SOCP is equivalent to QCQP
- If $A_i = 0$, $\forall i$, SOCP is equivalent to LP

Robust LP

Consider inequality constrained LP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \le b_i, \;\; i = 1, \ldots, m \end{array}$$

Parameters a_i are not accurate. They are only known to lie in given ellipsoids described by $\bar{a}_i$ and $P_i \in \mathbf{R}^{n \times n}$:

$$a_i \in \mathcal{E}_i = \{\bar{a}_i + P_i u | \|u\|_2 \le 1\}$$

Since $\sup\{a_i^T x | a_i \in \mathcal{E}\} = \bar{a}_i^T x + \|P_i^T x\|_2$,

Robust LP (satisfy constraints for all possible a_i) formulated as SOCP:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \|P_i^T x\|_2 \le b_i, \;\; i = 1, \ldots, m \end{array}$$

Dual QCQP

Primal (convex) QCQP

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\ \text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, 2, \ldots, m \\ & Ax = b \end{array}$$

Lagrangian: $L(x, \lambda) = (1/2)x^T P(\lambda) x + q(\lambda)^T x + r(\lambda)$ where

$$P(\lambda) = P_0 + \sum_{i=1}^m \lambda_i P_i, \quad q(\lambda) = q_0 + \sum_{i=1}^m \lambda_i q_i, \quad r(\lambda) = r_0 + \sum_{i=1}^m \lambda_i r_i$$

Since $\lambda \succeq 0$, we have $P(\lambda) \succ 0$ if $P_0 \succ 0$ and

$$g(\lambda) = \inf_x L(x, \lambda) = -(1/2)q(\lambda)^T P(\lambda)^{-1} q(\lambda) + r(\lambda)$$

Lagrange dual problem:

$$\begin{array}{ll} \text{maximize} & -(1/2)q(\lambda)^T P(\lambda)^{-1} q(\lambda) + r(\lambda) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

KKT Conditions for QP

Primal (convex) QP with linear equality constraints:

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Ax = b \end{array}$$

KKT conditions:

$$Ax^* = b, \qquad Px^* + q + A^T \nu^* = 0$$

which can be written in matrix form:

$$\begin{bmatrix} P & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x^* \\ \nu^* \end{bmatrix} = \begin{bmatrix} -q \\ b \end{bmatrix}$$

Solving a system of linear equations is equivalent to solving equality constrained convex quadratic minimization

Monomials and Posynomials

Monomial as a function $f : \mathbf{R}^n_+ \to \mathbf{R}$:

$$f(x) = d x_1^{a^{(1)}} x_2^{a^{(2)}} \ldots x_n^{a^{(n)}}$$

where the multiplicative constant $d \geq 0$ and the exponential constants $a^{(j)} \in \mathbf{R}, j = 1, 2, \ldots, n$

Sum of monomials is called a posynomial:

$$f(x) = \sum_{k=1}^{K} d_k x_1^{a_k^{(1)}} x_2^{a_k^{(2)}} \ldots x_n^{a_k^{(n)}}.$$

where $d_k \geq 0,\ k = 1, 2, \ldots, K$, and $a_k^{(j)} \in \mathbf{R},\ j = 1, 2, \ldots, n, k = 1, 2, \ldots, K$

Example: $\sqrt{2} x^{-0.5} y^{\pi} z$ is a monomial, $x - y$ is not a posynomial

GP

- GP standard form with variables x:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 1, \;\; i = 1, 2, \ldots, m, \\ & h_l(x) = 1, \;\; l = 1, 2, \ldots, M \end{array}$$

where $f_i, i = 0, 1, \ldots, m$ are posynomials and $h_l, l = 1, 2, \ldots, M$ are monomials

Log transformation: $y_j = \log x_j, b_{ik} = \log d_{ik}, b_l = \log d_l$

- GP convex form with variables y:

$$\begin{array}{ll} \text{minimize} & p_0(y) = \log \sum_{k=1}^{K_0} \exp(a_{0k}^T y + b_{0k}) \\ \text{subject to} & p_i(y) = \log \sum_{k=1}^{K_i} \exp(a_{ik}^T y + b_{ik}) \leq 0, \;\; i = 1, 2, \ldots, m, \\ & q_l(y) = a_l^T y + b_l = 0, \;\; l = 1, 2, \ldots, M \end{array}$$

In convex form, GP with only monomials reduces to LP

Convexity of LogSumExp

Log sum inequality (readily proved by the convexity of $f(t) = t \log t, t \geq 0$):

$$\sum_{i=1}^{n} a_i \log \frac{a_i}{b_i} \geq \left(\sum_{i=1}^{n} a_i \right) \log \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}$$

where $a_i, b_i \in \mathbf{R}_+, i = 1, 2, \ldots, n$

Let $\hat{b}_i = \log b_i$ and $\sum_{i=1}^{n} a_i = 1$:

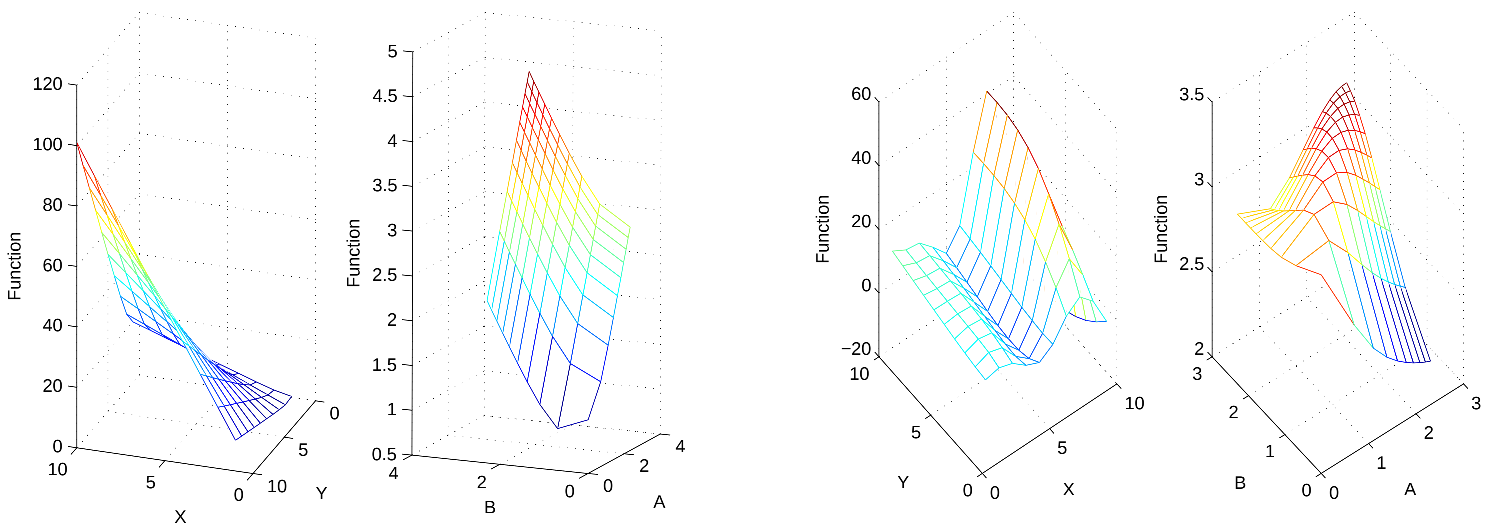
$$\log \left(\sum_{i=1}^{n} e^{\hat{b}_i} \right) \geq a^T \hat{b} - \sum_{i=1}^{n} a_i \log a_i$$

So LogSumExp is the conjugate function of negative entropy

Since all conjugate functions are convex, LogSumExp is convex

GP and Convexity

Convexity is not invariant under nonlinear change of coordinates



Extensions of GP

- The following problem can be turned into an equivalent GP:

$$\begin{array}{ll} \text{maximize} & x/y \\ \text{subject to} & 2 \leq x \leq 3 \\ & x^2 + 3y/z \leq \sqrt{y} \\ & x/y = z^2 \end{array}$$

$$\begin{array}{ll} \text{minimize} & x^{-1}y \\ \text{subject to} & 2x^{-1} \leq 1, \;\; (1/3)x \leq 1 \\ & x^2y^{-1/2} + 3y^{1/2}z^{-1} \leq 1 \\ & xy^{-1}z^{-2} = 1 \end{array}$$

- Let p, q be posynomials and r monomial

$$\begin{array}{ll} \text{minimize} & p(x)/(r(x) - q(x)) \\ \text{subject to} & r(x) > q(x) \end{array}$$

which is equivalent to

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & p(x) \leq t(r(x) - q(x)) \\ & (q(x)/r(x)) < 1 \end{array}$$

which is in turn equivalent to

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & (p(x)/t + q(x))/r(x) \leq 1 \\ & (q(x)/r(x)) < 1 \end{array}$$

- Generalized posynomials: composition of two posynomials.

Generalized GP: minimize generalized posynomials over upper bound inequality constraints on other generalized posynomials

Generalized GP can be turned into equivalent GP

Dual GP

Primal problem: unconstrained GP in variables y

$$\text{minimize} \quad \log \sum_{i=1}^{N} \exp(a_i^T y + b_i).$$

Lagrange dual in variables ν:

$$\begin{array}{ll} \text{maximize} & b^T\nu - \sum_{i=1}^N \nu_i \log \nu_i \\ \text{subject to} & \mathbf{1}^T\nu = 1, \\ & \nu \succeq 0, \\ & A^T\nu = 0 \end{array}$$

Dual GP

Primal problem: General GP in variables y

$$\begin{array}{ll} \text{minimize} & \log \sum_{j=1}^{k_0} \exp(a_{0j}^T y + b_{0j}) \\ \text{subject to} & \log \sum_{j=1}^{k_i} \exp(a_{ij}^T y + b_{ij}) \leq 0, \quad i = 1, \ldots, m, \end{array}$$

Lagrange dual problem:

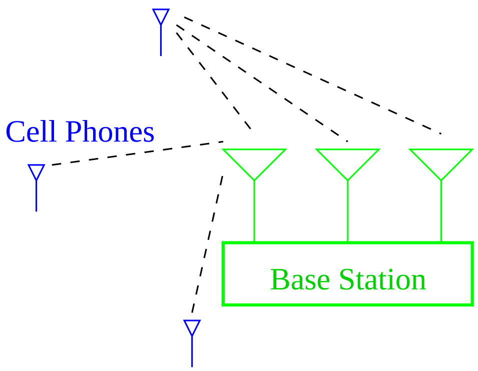
$$\begin{array}{ll} \text{maximize} & b_0^T \nu_0 - \sum_{j=1}^{k_0} \nu_{0j} \log \nu_{0j} + \sum_{i=1}^{m} \left(b_i^T \nu_i - \sum_{j=1}^{k_i} \nu_{ij} \log \frac{\nu_{ij}}{\mathbf{1}^T \nu_i} \right) \\ \text{subject to} & \nu_i \succeq 0, \quad i = 0, \ldots, m, \\ & \mathbf{1}^T \nu_0 = 1, \\ & \sum_{i=0}^{m} A_i^T \nu_i = 0 \end{array}$$

where variables are $\nu_i, \;\; i = 0, 1, \ldots, m$

A_0 is the matrix of the exponential constants in the objective function, and $A_i, \;\; i = 1, 2, \ldots, m$ are the matrices of the exponential constants in the constraint functions

Wireless Network Power Control

Wireless CDMA cellular or multi-hop networks:



Competing users all want:

- O1: High data rate
- O2: Low queuing delay
- O3: Low packet drop probability due to channel outage

Optimize over transmit powers $\mathbf{P}$ such that:

- O1, O2 or O3 optimized for 'premium' QoS class (or for maxmin fairness)
- Minimal QoS requirements on O1, O2 and O3 met for all users

A Sample of Power Control Problems

2 classes of traffic traverse a multihop wireless network:

maximize	Total System Throughput
subject to	Data $\text{Rate}_1 \geq$ Rate Requirement_1
	Data $\text{Rate}_2 \geq$ Rate Requirement_2
	Queuing $\text{Delay}_1 \leq$ Delay Requirement_1
	Queuing $\text{Delay}_2 \leq$ Delay Requirement_2
	Drop $\text{Prob}_1 \leq$ Drop Requirement_1
	Drop $\text{Prob}_2 \leq$ Drop Requirement_2
variables	Powers

Wireless Channel Models

Signal Interference Ratio:

$$\mathsf{SIR}_i(\mathbf{P}) = \frac{P_i G_{ii}}{\sum_{j \neq i}^{N} P_j G_{ij} + n_i}.$$

Attainable data rate at high SIR:

$$c_i(\mathbf{P}) = \frac{1}{T} \log_2(K \mathsf{SIR}_i(\mathbf{P})).$$

Outage probability on a wireless link:

$$P_{o,i}(\mathbf{P}) = \mathbf{Prob}\{\mathsf{SIR}_i(\mathbf{P}) \leq \mathsf{SIR}_{th}\}$$

Average (Markovian) queuing delay with Poisson(Λ_i) arrival:

$$\bar{D}_i(\mathbf{P}) = \frac{1}{\Gamma c_i(\mathbf{P}) - \Lambda_i}$$

Conversion to GP

- Maximize the system throughput equivalent to minimize posynomial $\prod_i \mathsf{ISR}_i$
- Extension: maximize weighted sum of data rates: $\sum_i w_i R_i$
- Outage probability bound:

$$\begin{aligned} P_{o,i} &= \mathbf{Prob}\{\mathsf{SIR}_i \leq \mathsf{SIR}_{th}\} \\ &= 1 - \prod_{k \neq i} \frac{1}{1 + \frac{\mathsf{SIR}_{th} G_{ik} P_k}{G_{ii} P_i}} \end{aligned}$$

- Delay bound:

$$\begin{aligned} \frac{1}{\frac{\Gamma}{T}\log_2(\mathsf{SIR}_i) - \Lambda_i} &\leq \bar{D}_{i,max} \\ \mathsf{ISR}_i(\mathbf{P}) &\leq 2^{-\frac{T}{\Gamma}(\bar{D}_{max}^{-1} + \Lambda_i)} \end{aligned}$$

Cellular Case

$$
\begin{array}{ll}
\text{maximize} & \mathsf{SIR}_{i*} \\
\text{subject to} & \mathsf{SIR}_k \geq \mathsf{SIR}_{k,min}, \;\; \forall k, \\
 & \sum_{j \in I_k} P_j d_j^{-\gamma_j} \alpha_j < c_k, \;\; \forall k, \\
 & P_{k1} G_{k1} = P_{k2} G_{k2}, \\
 & P_k \leq P_{k,max}, \;\; \forall k, \\
 & P_k \geq 0, \;\; \forall k.
\end{array}
$$

Recovers classical near-far problem in CDMA

Max-min fairness objective function:

$$
\text{maximize} \;\; \min_k \mathsf{SIR}_k
$$

GP Formulations

Any combination is GP in high SIR

Any combination without (C,D,c) is GP in any SIR

Objective Function	*Constraints*
(A) Maximize R_{i^*}	(a) $R_i \geq R_{i,min}$
(B) Maximize $\min_i R_i$	(b) $P_{i1}G_{i1} = P_{i2}G_{i2}$
(C) Maximize $\sum_i R_i$	(c) $\sum_i R_i \geq R_{system,min}$
(D) Maximize $\sum_i w_i R_i$	(d) $P_{o,i} \leq P_{o,i,max}$
(E) Minimize $\sum_i P_i$	(e) $0 \leq P_i \leq P_{i,max}$

Power Control by GP

This suite of nonlinear nonconvex power control problems can be solved by GP (in standard form)

- Global optimality obtained efficiently
- For many combination of objectives and constraints
- Multi-rate, Multi-class, Multi-hop
- Feasibility $\Rightarrow$ Admission control
- Reduction in objective $\Rightarrow$ Admission pricing

Key advantage: Allows nonlinear objectives and seemingly nonconvex constraints

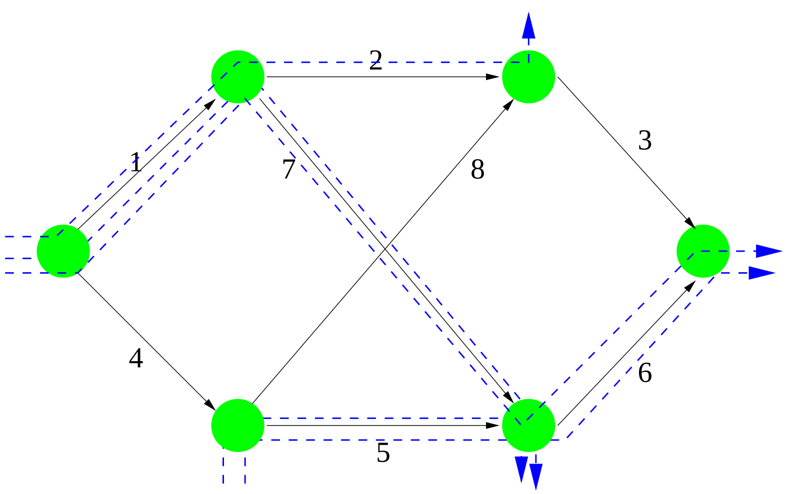
Key idea: These functions of SIR can be written as posynomials in $\mathbf{P}$

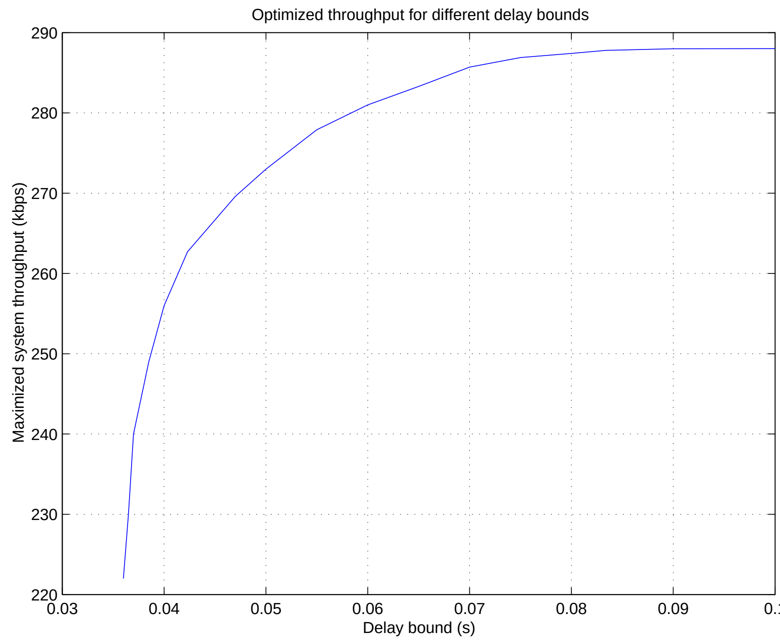
New results: low SIR regime and distributed algorithms

Numerical Example: Optimal Throughput-Delay Tradeoff

- 6 nodes, 8 links, 5 multi-hop flows, Direct Sequence CDMA
- max. power 1mW, target BER 10^{-3}, path loss $=$ distance^{-4}

Maximized throughput of network increases as delay bound relaxes





Heuristics: Delay bound violation or system throughput reduction

Lecture Summary

- First type of nonlinearity: quadratic.
- Second type of nonlinearity: Posynomial or LogSumExp.
- Nonlinear problems that are, or can be converted into, convex optimization: QCQP (SOCP) and GP. Both cover LP as special cases.
- Significantly extend scope of optimization-theoretic modelling, analysis, and design.

Readings:

- Section 4.4, 4.5, 5.5, 5.7 in Boyd and Vandenberghe
- M. Chiang, "Geometric Programming for Communication Systems", *Trends and Foundations in Information and Communications Theory*, vol. 2, no. 1, pp. 1-156, 2005.